Optimal Staking and Liquid Token Holding Decisions in Cryptocurrency Markets

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Abstract

We explore an optimal token holding and staking problem for cryptocurrency investors. Our investigation revolves around understanding the tradeoff between staking rewards/utility and the consequent illiquidity that emerges as a result of investor heterogeneity and the distinct structure of blockchain platforms or Decentralized Autonomous Organizations (DAOs). We present comprehensive analytic solutions, which enable us to examine the novel facets and implications stemming from the staking mechanism for trading and staking policies and the dynamics of risk-taking behaviors. These insights extend beyond token investments devoid of staking rewards and conventional investment avenues, such as stocks and commodities.

Keywords: Cryptocurrency Investment, Staking, DAO (Decentralized Autonomous Organization), Blockchain, Proof of Stake, DeFi (Decentralized Finance).

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1. Introduction

Staking presents a distinctive and captivating aspect of cryptocurrency investment that sets it apart from traditional assets such as stocks, bonds, and commodities. Typically, investors engage in staking for two primary reasons: (i) to generate additional income and/or (ii) to actively contribute to the decision-making processes of a blockchain platform. With regard to the first purpose, the staking *reward* commonly involve compensating block validators within a platform that operates on the Proof of Stake (POS) mechanism. However, given the proliferation of Defi (Decentralized Finance) applications, yield farming protocols, and even Centralized Exchanges (CEXs), individual investors can now readily stake their tokens through delegation services offered by these platforms to earn supplementary income. In terms of active participation, the rise of Decentralized Autonomous Organizations (DAOs) has been notable.[1] DAOs primarily function as decentralized entities making governance and investment decisions. These decisions are typically determined through voting among investors who stake governance tokens on the platform (see Section 2.2 for more details). Stakers, or voters, are often enthusiastic investors who derive *utility* from participating in the decision-making processes of the DAO.

It is important to consider that staked tokens while providing rewards or utility through active investing, come with the drawback of illiquidity associated with frictions and costs inherent in blockchain platforms or DAO (e.g., lock-up periods and unlocking fees). Consequently, the tradeoff between the benefits (reward/utility) and the illiquidity has a substantial influence on the decision between holding tokens in a liquid account versus staking them in a locked account. This paper aims to formalize the problem faced by a cryptocurrency investor (or agent interchangeably) in navigating this tradeoff and to investigate the agent's optimal trading strategies and their novel implications arising from the staking mechanism.

To accomplish this, we expand upon a standard continuous-time model of the agent's consumption and portfolio selection problem by introducing the tradeoff. The agent's wealth consists of three components: cash, liquid token holdings, and staked token holdings. Following the framework of the utility platform literature (e.g., Cong et al. (2022b, 2021b) and Cong et al. (2021a)), we also assume that the agent can derive utility not only from consumption but also from staking tokens. More specifically, we consider heterogeneity in the agent's preference indexed by the relative weight on the utility from staking tokens. The agent is called an active participant if the weight is high and a general investor if the weight is low. For example, active participants include initial founders, members of the DAO team, early participants, and blockchain developers. On the contrary, general investors have zero or relatively low utility

[1] Insert here about the total market cap of DAOs from the CoinMarketcap data as of June 2023.

from staking and thus primarily engage in staking to earn extra income.

Our theoretical contribution is that the solution analysis introduces a novel approach by utilizing the duality principle to formulate the dual problem with two singular controls. One control governs the trajectory of the number of staked tokens, while the other enforces the nonnegativity of wealth (in compliance with the borrowing constraint). We demonstrate that the highest dual value achievable through staking decisions, given any strategies satisfying the borrowing constraint (maxmin), is equal to the lowest dual value enforced by the borrowing constraint, given any staking strategies (minmax). This equivalence between the maxmin and minmax operations implies that the agent's problem can be interpreted as a zero-sum game aimed at achieving the dual value.

We derive fully analytic solutions, uncovering the existence of an inaction interval where the agent does not change their staked position (i.e., neither stake additional tokens nor unlock the staked tokens). Whenever the ratio of wealth to the value of currently staked tokens reaches a threshold level, the agent stakes additional tokens. Conversely, when wealth reaches zero, the agent's behavior depends on the cost of unlocking staked tokens. If the cost is low, the agent optimally liquidates staked tokens to prevent wealth from depleting to zero. If the cost is high, unlocking the staked tokens is never optimal. Instead, the agent engages in short-selling of liquid tokens to offset the staked token position, ensuring that total wealth never reaches zero.[2]

The model provides significant insights into optimal staking behaviors. First, we find that the frequency of changing the staked token position increases as the cost decreases or the reward increases. A lower cost allows for more frequent position adjustments, while a higher staking reward incentivizes greater token staking. Furthermore, an intriguing finding regarding frequency is that active DAO participants change their staked position more frequently than general investors. This result may initially seem counterintuitive since active DAO participants resemble long-term investors, and as such, they are expected to maintain a sticky staked position. However, active participants derive greater utility gains from staking, making them more inclined to increase the number of staked tokens in response to even small increases in wealth. Therefore, the frequency of changing the staking position increases as the agent gains greater utility from staking.[3]

Second, let us explain the optimal investment and consumption strategy. As the token price increases within the inaction region, the optimal decision is to increase both consump-

[2]Note that short selling of cryptocurrency tokens is available on many cryptocurrency exchanges through perpetual futures contracts or margin requirements.

[3]We perform simple empirical tests to verify this result by using the onchain data of the DAO platforms such as Maker DAO, Aave DAO, and Lido DAO (see Section A of the Appendix).

tion and liquid token holdings. However, the ratio of consumption to wealth decreases as wealth increases, while the ratio of liquid token holdings to wealth increases. Essentially, as wealth increases, the agent becomes more inclined to take on additional risk. There are two underlying factors contributing to this result: (i) as wealth increases, the borrowing constraint becomes less restrictive, allowing for greater risk-taking, and (ii) by increasing investment in liquid tokens, there is a higher possibility of reaching the upper boundary, leading to higher future rewards.

Third, this propensity for risk-taking bears significant dynamic implications for individual token holdings. While in the standard Merton model the ratio of risky asset holdings to total wealth is constant, our findings diverge in the case of investing tokens with staking: the ratio of risky token holdings to total wealth can be notably higher during good times compared to bad times, owing to the risk-taking effect spurred by the staking mechanism. This noteworthy revelation underscores the distinct investment patterns between individual cryptocurrency investors, particularly stakers and non-stakers. Precisely speaking, non-staking is not considered optimal given the presence of a staking reward. A non-staker in our context refers to an investor who is unable to stake the tokens by any reason. For example, this investor exclusively trades the cryptocurrency in a centralized exchange that does not provide staking services for the token. In this case, stakers exhibit a tendency to hold substantially higher token amounts in good times, while they hold relatively fewer token amounts in bad times when contrasted with non-stakers' holdings.

Forth, we discover that the impact of the staking reward differs significantly depending on the type of investor. For general investors, an increase in the staking reward results in an increase in the staking ratio (i.e., the proportion of staked tokens out of total token holdings). However, for active participants, an increase in the staking reward leads to a decrease in the staking ratio. To comprehend this seemingly counterintuitive finding, it is important to note that staking provides the agent with an additional income stream, making the problem analogous to a standard consumption-investment problem with stochastic income. When the staking reward increases, the agent's total wealth effectively rises, which holds true for all types of investors. The difference lies in the allocation of the increased total token holdings. For general investors, the increase in staked tokens outweighs the increase in liquid token holdings, while the opposite is true for active investors. As the staking reward increases, active participants, who have a relatively higher marginal utility from staking than from consumption, exhibit higher risk aversion, implying that unlocking staked tokens incurs a greater utility loss. Hence, active participants decrease the staking ratio while the total token holdings increase, driven by the income effect.

Finally, we explore an extended model wherein the agent engages in both the cryptocurrency market with staking and an additional risky asset, such as stocks or other cryptocurrencies that lack a staking mechanism. This consideration grows in significance due to the rapid growth of cryptocurrency's share in households' total portfolios worldwide.[4] We present the analytical solution for the extended model, along with its associated implications. Firstly, the staking ratio may vary, either higher or lower, when the agent invests in both markets compared to investing in a single cryptocurrency. This variation depends on the characteristics of the investors and the wealth effect. Secondly, and of particular interest, the correlation between the staking cryptocurrency and the additional risky asset is important. The ratio of the total asset holdings to the staking position diminishes as the correlation between the two assets increases. This finding holds significance considering that the majority of cryptocurrencies exhibit relatively high correlations with each other. Consequently, investors holding staking tokens primarily tend to possess corresponding tokens rather than another non-staking cryptocurrencies.

Literature review: Our paper is situated within two different streams of literature: the classical consumption-investment literature and the rapidly expanding literature on the POS mechanism. In relation to the consumption-investment literature, our theoretical framework builds upon two types of models. Firstly, we build upon the models that incorporate illiquid assets or durable goods, such as those proposed by Grossman and Laroque (1990), Cuoco and Liu (2000), Flavin and Nakagawa (2008), Dai et al. (2011), and Chetty and Szeidl (2016). Secondly, we extend the models that consider transaction costs, as examined by Davis and Norman (1990), Shreve and Soner (1994), Cadenillas (2000), Jang et al. (2007), and Dai et al. (2015). The former class of models investigates investment involving two different types of assets: liquid assets, like stocks, which have no trading friction, and illiquid assets, like housing, which have trading frictions. The latter class exclusively focuses on stock trading with transaction frictions. In contrast, our model explores cryptocurrency investment, which introduces the unique element of staking in addition to holding liquid tokens. Consequently, investors must determine the optimal allocation between liquid and illiquid positions for the same asset. Therefore, our model captures this distinctive feature of cryptocurrency investment.

The POS literature includes notable contributions from Cong et al. (2022a), John et al. (2022), Rosu and Saleh (2021), and Saleh (2021). Their primary focus is on various aspects of POS, such as determining the equilibrium staking reward (Cong et al. (2022a) and John

[4]For example, according to Weber et al. (2023) who analyzed survey data from roughly 80,000 U.S. households, the proportion of individuals owning any cryptocurrency significantly increased between 2021 and 2022 from approximately 3% to 11%.

et al. (2022)) or identifying the conditions for achieving the POS consensus (Rosu and Saleh (2021) and Saleh (2021)). In contrast, our research diverges from the existing literature by investigating the impact of the tradeoff between the staking reward and illiquidity on individual investors' portfolio choices. We consider the presence of heterogeneity among investor types and staking frictions, which are determined by the specific characteristics of blockchain platforms or decentralized autonomous organizations (DAOs).

Technically, our paper builds upon the framework of singular control problems, as explored by Abel and Eberly (1996), Miao and Zhang (2015), Choi et al. (2022), and Choi et al. (2023a). Additionally, we extend the analysis of the zero-sum game problem, as examined by Farhi and Panageas (2007).[5] In our problem, we encounter two singular controls, and the min-max and maxmin problems associated with determining the operational order of these two controls share a similar nature to a zero-sum game problem. To address this challenge, we develop a novel duality method that enables us to effectively handle problems involving two singular controls.

The rest of the paper proceeds as follows. Section 2 provides the model. The solution analysis and the optimal policies are presented in Section 3. Section 4 investigates various implications of the optimal policies. In Section 5 we extend the model to the case where there is an additional risky asset such as a cryptocurrency with no staking mechanism or a stock. Section 6 provides concluding remarks. Appendix A provides simple empirical results. All the proofs are in Appendix B.

2. Model

We consider an agent's problem of optimal consumption and investment in the cryptocurrency market. We first describe the mathematical setup of the model in Section 2.1. Then, in Section 2.2, we provide a detailed description of the model, including how to interpret the key parameters of the model.

2.1. Setup

A platform (or a DAO) issues a cryptocurrency token. Assume that the price dynamics of the token is given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

[5] The zero-sum game interpretation of the optimal portfolio selection and stopping time problem is exclusively found in the Online Appendix of Farhi and Panageas (2007).

where B_t is a standard Brownian motion in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, μ is the expected growth rate of the token price, and σ is its volatility. μ and σ are positive constants. We will provide a discussion for the token usages or rights in Section 2.2.

At each time t, the agent determines its consumption level and the allocation of funds between tokens and a risk-free asset that provides a constant rate of return $r > 0$. Additionally, besides holding tokens, the agent has the option to stake some of the tokens on the platform. Staking serves two purposes: acquiring governance rights and earning additional income. The relative strength of these motivations depends on the agent's preferences, which will be explained in detail later. With regard to staking rewards, the agent can either stake directly on the platform or delegate their tokens to staking service providers.[6]

Consider x_t as the number of tokens held by the agent (not staked), and let k_t represent the number of tokens staked at time t. Therefore, the agent's total token holdings can be defined as $\pi_t := x_t + k_t$. The dynamics of k_t is given by

$$dk_t = -\delta k_t dt + d\mathcal{G}_t^+ - d\mathcal{G}_t^-, \tag{1}$$

where $\mathcal{G}_t^+$ (or $\mathcal{G}_t^-$) is the cumulative amount of tokens being staked (unlocked) at t and $\delta > 0$ represents the depreciation rate of the staked tokens by being slashed, hacked, burned, or paid for the management fee of the DAO (see Section 2.2.3.) The agent earns rewards over time that are proportional to its staking amount. Specifically, the instantaneous staking reward is expressed as $\phi k_t S_t dt$, where ϕ is a constant (see Section 2.2.1). Conversely, there is a cost associated with unlocking the staked tokens, denoted as $\rho S_t d\mathcal{G}_t^-$, where the constant $\rho \in [0, 1)$ represents the token deduction rate per unit of token withdrawal (see Section 2.2.2). It is assumed, without loss of generality, that there is no cost when locking up tokens for staking.[7]

Let W_t be the agent's wealth at t. W_t consists of three components: liquid token holdings, risk-free assets, and staked (or illiquid) token holdings. Then, these three components determine the wealth dynamics dW_t as follows with an initial with $W_0 = w_0$:

$$\begin{aligned} dW_t &= \pi_t dS_t + r(W_t - \pi_t S_t)dt + \phi k_t S_t dt - S_t d\mathcal{G}_t^+ + S_t(1-\rho)d\mathcal{G}_t^- - c_t dt \\ &= [rW_t + (\mu - r)\pi_t S_t + \phi k_t S_t - c_t]dt - S_t d\mathcal{G}_t^+ + (1-\rho)S_t d\mathcal{G}_t^- + \sigma \pi_t S_t dB_t, \end{aligned} \tag{2}$$

where $\pi_t dS_t$ is the instantaneous change of the value of the total token holdings, $r(W_t - \pi_t S_t)dt$ is the change of the risk-free asset holdings, $\phi k_t S_t dt$ is the instantaneous reward from the

[6] For example, StakeWise, StakeWithUs, Stakin, Staking Facilities, Stake.fish, and most cryptocurrency exchanges provide staking services.

[7] Alternatively, it could be assumed that there is a cost when locking up tokens, as long as the cost of unlocking is higher than the cost of locking, thereby preserving the main results.

staked token holdings, $c_t dt$ is instantaneous consumption, and $\rho S_t d\mathcal{G}_t^-$ is the cost from unlocking tokens.

Finally, the agent's problem is stated as follows. Given the initial wealth $W_0 = w$, the initial staking $k_0 = k$, and the initial token price $S_0 = s$, the agent maximizes his/her expected utility by choosing consumption (c_t), liquid token holdings (x_t), and staking decisions ($\mathcal{G}_t^+$ and $\mathcal{G}_t^-$). More precisely, for $\omega \in (0, 1]$,

$$V(w, k, s) := \max_{(c_t, x_t, \mathcal{G}_t^+, \mathcal{G}_t^-)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\omega u_1(c_t) + (1-\omega)u_2(k_t S_t)\right) dt\right] \tag{3}$$

with $(W_0, k_0, S_0) = (w, k, s)$ subject to the budget constraints (2) and the borrowing constraint:

$$W_t \geq 0 \quad \text{for all} \quad t \geq 0, \tag{4}$$

where $\beta > 0$ is the subjective discount factor and ω is the weights between utilities from consumption and staking. In our model, we suppose $u_1(z) = u_2(z) = u(z)$ with

$$u(z) = \frac{z^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \ \gamma \neq 1.$$

Note that the agent with $\omega < 1$ has utility $(1-\omega)u(k_t S_t)$. This can be interpreted as the utility gains obtained from transacting on the platform, which is commonly assumed in utility token platform models (e.g., Cong et al. (2022b, 2021b) and Sockin and Xiong (2023)). In the case of a DAO, we can alternatively assume that the agent also derives utility from staking tokens, as staking enables participation in voting and various governance decisions of the DAO, including investment decisions. The utility increases with the total value of staking, $k_t S_t$, as a higher stake corresponds to greater voting power.

2.2. Background

This section offers detailed information on our modeling approach. Understanding the model hinges on the staking mechanism utilized by the blockchain platforms along with the objectives, structure, and functioning of the DAOs. Notably, the sign and magnitude of key parameters dictate the characteristics of both the platforms and the DAOs. Before we delve into a detailed explanation of the three essential parameters, let us initially address the following two preliminary aspects.

Firstly, an increasing number of blockchain platforms are adopting Proof of Stake (POS) as the underlying consensus mechanism instead of Proof of Work (POW).[8] In the POS mechanism, block validators are required to stake tokens, similar to miners in the POW mechanism.

[8]This shift stems from the high transaction costs associated with POW and the widespread criticism of the substantial fossil energy consumption it entails.

Rewards comprise two components: block rewards and transaction fees paid by customers when executing smart contracts for various purposes. In our model, we do not differentiate between block rewards and transaction fees. Instead, we assume that during each short time interval $[t, t + dt)$, an investor receives a total of $\phi k_t dt$ tokens. Furthermore, in some cases, even if a platform does not employ the POS mechanism, investors can still earn yields by staking tokens through specific services such as flash loans, collateralized lending, and CEXs. Therefore, thanks to the development of various DeFi services, investors can earn rewards by staking tokens not only from POS-based platforms but also from POW-based platforms and other consensus mechanisms.

Secondly, certain DAOs grant governance rights to token stakers, similar to the equity ownership rights associated with stock holdings. As a result, stakers in a DAO receive tokens as rewards generated by the tasks undertaken by the DAO or the projects in which the DAO invests. This type of DAO resembles a conventional investment club, with a crucial distinction being that a DAO operates through decentralized decision-making processes on the blockchain, where investment decisions are made through voting by token stakers. The Ethereum DAO in 2015 was the pioneering example of such a setup. In this context, investment proposals are submitted and approved by voting of the staker, after which stakers receive returns from the investments. Since then, DAOs have focused primarily on on-chain investments, including NFT trading, cryptocurrency investments, or ICOs (Initial Coin Offerings) within the DeFi ecosystem. However, more recently, DAOs that specialize in off-chain investments have emerged. For example, since the approval of Wyoming's SF0038 legislation, several offline-based DAOs have emerged (`https://www.wyoleg.gov/Legislation/2021/SF0038`). These DAOs, like investment clubs, aim to invest in various assets, including physical and virtual assets, by leveraging diversification and risk-sharing principles, as well as relying on the wisdom of the crowd. Given this context, we assume that the DAO generates a constant flow of investment returns denoted by ϕ, which is distributed to the stakers.

With these two scenarios in mind, we provide more detailed explanations of the important model parameters in Sections 2.2.1, 2.2.2, and 2.2.3.

2.2.1. Staking reward

The condition $\phi > 0$ simply signifies that the agent receives a reward by staking tokens in a POS staking pool or by participating in an investment club-like DAO.

It is important to note that the staking reward $\phi k_t S_t dt$ directly affects the budget dynamics in Equation (2). This indicates that the reward is provided in the form of liquid tokens. However, in certain staking pools, the reward tokens may not be withdrawable immediately and may be subject to a lock-up period. The agent has the option to stake tokens directly on the platform through a smart contract designed for staking. Alternatively, there are indirect methods of staking through delegators, such as decentralized applications for staking and yield farming and centralized exchanges. Many investors opt for these service providers despite paying service fees. In the case of using such services, the reward rate ϕ may be lower than the actual reward rate offered by direct on-chain staking. However, this consideration primarily has a quantitative impact and does not alter the qualitative findings of the model. It is important to note that indirect stakers are unable to participate in the decision-making processes of the DAO. In other words, these indirect stakers are general investors who do not derive utility value from holding tokens in our model.

2.2.2. Cost from unlocking the staked tokens

The condition $\rho > 0$ indicates the presence of costs associated with inconveniences, illiquidity, gas fees, or disincentives when unlocking staked tokens. These costs arise due to factors such as the existence of a locking period. To aid in the clarity of the explanation, let us consider the following three cases:

A. In the context of individual on-chain staking, smart contracts strictly enforce a locking period. This means that investors are unable to unlock their tokens until the designated locking period has elapsed.

B. However, it is important to note that many recent DeFi staking protocols and CEXs offer staking services that allow investors to unlock tokens before the enforced lock-up period ends. This early unlocking option is possible because these providers manage their own token inventory and pool customers' tokens. Instead of preventing early unlocking outright, these providers impose a penalty fee against unlocked tokens ahead of time.

C. Additionally, it is worth highlighting that Proof of Stake (POS) protocols featuring lengthy locking periods often give rise to a market for staked tokens, where they are traded at a discount in comparison to liquid tokens. As an illustration, during Ethereum's transition from Proof of Work (POW) to POS, the price of staked Ethereum was consistently 1-9% lower than that of regular Ethereum. This price difference was attributed to the inability to unlock staked tokens within the migration period (see: https://www.bitstamp.net/markets/eth2/eth/). In this case, ρ can be interpreted as the price disparity between liquid tokens and illiquid staked tokens.

Note that our model primarily focuses on Cases B and C mentioned above rather than Case A. In Case A, the gas fees associated with locking and unlocking are the same. However, in Cases B and C, locking usually involves zero fees, and if applicable, these fees are much smaller compared to unlocking. We believe that the prominence of Cases B and C has grown among investors, largely attributed to the rapid expansion of the Defi market.

2.2.3. Depreciation: Slashing, Hacking, Burning, or Management fees

The depreciation rate of the staked tokens, indicated as $\delta > 0$, can have different implications depending on the nature of the DAO. In POS platforms, a slashing mechanism is often implemented to discourage any misbehavior from stakers. Slashing may be required as part of the consensus mechanism, even if a staker has no intention of misbehaving. For instance, in certain protocols, stakers are responsible for posting instantaneous prices of specific tokens from different exchanges during each block validation. If a posted price significantly deviates from the mean or median value of all postings, the corresponding staker may be penalized, resulting in a portion of their staked tokens being slashed. Burning of staked tokens can also occur in a DAO as a result of voting. Stakers can also experience token loss due to security attacks or storage failures, as the IP address of a staker is often exposed for an extended period within the network. It should be noted that δ can also be interpreted as managerial or operational expenses, particularly for off-chain DAOs. In this case, the expenses are typically shared by stakers in proportion to their staking amounts. Costs may increase linearly with the number of tokens that are staked. For example, such costs could include hiring specialists like lawyers, accountants, or realtors if the DAO invests in physical assets. Tokenizing these assets requires proper offline registry services. As the DAO continuously buys and sells off-chain assets over time, registry services are necessary and incur associated costs.

In summary, we consider all the scenarios of losing staked tokens due to slashing, hacking, burning, and management fees as depreciation. As the cumulative impact of these possibilities is typically low, the result is a gradual depreciation of staked tokens over time.

3. Analysis

In this section, our approach begins by deriving a dual singular control problem corresponding to the primal Problem (3). Subsequently, we obtain the explicit solution to the dual problem. By utilizing the duality relationship, we recover the value function and, in turn, obtain the optimal policies.

3.1. Singular Control Problem

Let $H_t \equiv e^{-rt}\zeta_t$, where $\zeta_t = e^{-\frac{1}{2}\theta^2 t - \theta B_t}$ with $\theta \equiv (\mu - r)/\sigma$. The wealth process in (2) is rewritten as the static budget constraint as follows:

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty H_t D_t c_t dt\right] + \underbrace{\mathbb{E}\left[\int_0^\infty H_t D_t S_t d\mathcal{G}_t^+\right]}_{(i)} \\
&\qquad \le w_0 + \underbrace{\mathbb{E}\left[\int_0^\infty H_t D_t \phi k_t S_t dt + \int_0^\infty H_t D_t (1-\rho) S_t d\mathcal{G}_t^-\right]}_{(ii)},
\end{aligned} \tag{5}
$$

where $\{D_t\}_{t=0}^\infty$ is a positive, non-increasing, and right continuous with left limits process starting at 1. It is important to note that, in comparison to the static budget constraint typically considered in standard portfolio selection problems, constraint (5) in our case incorporates two additional terms, denoted as (i) and (ii), which capture the fundamental nature of the staking problem. Specifically, the left-hand side of inequality (5) represents the sum of the present value of the future consumption stream and the present value of money used for staking tokens over time (term (i)). On the other hand, the right-hand side represents the sum of the initial wealth and term (ii). Term (ii) consists of the present value of the reward streams generated by the staked tokens and the present value of money earned from unlocking the staked tokens.

Using (5) we set up Lagrangian $\mathfrak{L}$ as follows: for a Lagrangian multiplier $y > 0$

$$
\begin{aligned}
\mathfrak{L} :=& \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\omega u(c_t) + (1-\omega) u(k_t S_t)\right) dt\right] + y\left(w + \mathbb{E}\left[\int_0^\infty H_t D_t \phi k_t S_t dt\right]\right. \\
&\left. - \mathbb{E}\left[\int_0^\infty H_t D_t c_t dt + \int_0^\infty H_t D_t S_t d\mathcal{G}_t^+ - \int_0^\infty H_t D_t (1-\rho) S_t d\mathcal{G}_t^-\right]\right) \\
\le& \mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{\mathcal{G}_t^+, \mathcal{G}_t^-\}_{t=0}^\infty) + y w_0,
\end{aligned} \tag{6}
$$

where $\mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{\mathcal{G}_t^+, \mathcal{G}_t^-\}_{t=0}^\infty)$ is given by

$$
\begin{aligned}
\mathcal{J}(y, k, s; \{D_t\}_{t=0}^\infty, \{\mathcal{G}_t^+, \mathcal{G}_t^-\}_{t=0}^\infty) :=& \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\omega \tilde{u}\left(\frac{Y_t}{\omega}\right) + (1-\omega) u(k_t S_t) + \phi Y_t S_t k_t\right) dt\right. \\
&\left. - \int_0^\infty e^{-\beta t} Y_t S_t d\mathcal{G}_t^+ + (1-\rho)\int_0^\infty e^{-\beta t} Y_t S_t d\mathcal{G}_t^-\right]
\end{aligned} \tag{7}
$$

with $Y_t := y e^{\beta t} H_t D_t$, and

$$
\tilde{u}(z) := \sup_{c>0}\left(u(c) - yc\right) = \frac{\gamma}{1-\gamma} z^{-\frac{1-\gamma}{\gamma}}.
$$

We will derive a variational inequality using $\mathcal{J}$. To do so, first, we establish the following *weak-duality*:

$$V(w_0, s, k) \le \inf_{y>0} \sup_{\{\mathcal{G}_t^+, \mathcal{G}_t^-\}} \inf_{\{D_t\}} \left(\mathcal{J}(y, k, s; \{D_t\}_{t=0}^{\infty}, \{\mathcal{G}_t^+, \mathcal{G}_t^-\}_{t=0}^{\infty}) + y w_0 \right)$$
$$\le \inf_{y>0} \inf_{\{D_t\}} \sup_{\{\mathcal{G}_t^+, \mathcal{G}_t^-\}} \left(\mathcal{J}(y, k, s; \{D_t\}_{t=0}^{\infty}, \{\mathcal{G}_t^+, \mathcal{G}_t^-\}_{t=0}^{\infty}) + y w_0 \right) := J(y, k, s), \quad (8)$$

where $V(w_0, s, k)$ is the agent's value function defined by (3). Second, we will show that the inequality in (8) holds as equality, i.e.,

$$\sup_{\{\mathcal{G}_t^+, \mathcal{G}_t^-\}} \inf_{\{D_t\}} \mathcal{J}(y, k, s; \{D_t\}_{t=0}^{\infty}, \{\mathcal{G}_t^+, \mathcal{G}_t^-\}_{t=0}^{\infty}) = \inf_{\{D_t\}} \sup_{\{\mathcal{G}_t^+, \mathcal{G}_t^-\}} \mathcal{J}(y, k, s; \{D_t\}_{t=0}^{\infty}, \{\mathcal{G}_t^+, \mathcal{G}_t^-\}_{t=0}^{\infty}),$$

where we will denote this common value by $J(y, k, s)$ that is convex in y and the convexity is inherited by the construction of $\mathcal{J}$ in (7).

It's worth noting that the singular controls $\mathcal{G}_t^+$ and $\mathcal{G}_t^-$ determine the trajectory of the number of staked tokens k_t, while the control D_t ensures that the total wealth W_t remains non-negative (i.e., satisfying the borrowing constraint). Taking this into consideration, the above equality implies that the maximum dual value that can be achieved through staking decisions under any strategies that adhere to the borrowing constraint is equal to the minimum dual value imposed by the borrowing constraint, regardless of the staking strategies employed.

Now by dynamic programming principle, we first derive a Variational Inequality (VI) that J satisfies (see (B.1) in the Appendix). Note that the VI (B.1) is three dimensional, so we use the following transform to reduce the dimension.

$$\mathcal{Q}(z) := \frac{J(y, k, s)}{\omega (ks)^{1-\gamma}} \quad \text{with} \quad z = \frac{y (ks)^{\gamma}}{\omega}. \quad (9)$$

Then, we can rewrite (B.1) as the following variational inequality for $\mathcal{Q}(z)$: for $z > 0$

$$\begin{cases} \mathcal{L}_z \mathcal{Q} + \tilde{u}(z) + \tilde{\omega} u(1) + \phi z = 0 & \text{if } (1-\rho) z < (1-\gamma)\mathcal{Q} + \gamma z \mathcal{Q}' < z \text{ and } \mathcal{Q}' < 0, \\ \mathcal{L}_z \mathcal{Q} + \tilde{u}(z) + \tilde{\omega} u(1) + \phi z \le 0 & \text{if } (1-\gamma)\mathcal{Q} + \gamma z \mathcal{Q}' = z \text{ or } (1-\gamma)\mathcal{Q} + \gamma z \mathcal{Q}' = (1-\rho) z, \\ \mathcal{L}_z \mathcal{Q} + \tilde{u}(z) + \tilde{\omega} u(1) + \phi z \ge 0 & \text{if } \mathcal{Q}' = 0, \end{cases} \quad (10)$$

where the operator $\mathcal{L}_z$ is given by

$$\mathcal{L}_z := \frac{\sigma_z^2}{2} z^2 \frac{d^2}{dz^2} + (\beta_z - r_z) z \frac{d}{dz} - \beta_z,$$

where the coefficients are defined as follows: $\tilde{\omega} := (1-\omega)/\omega$, $r_z := r - (\mu - \delta) + \sigma\theta = \delta$, $\beta_z := \beta - (1-\gamma)(\mu - \delta) + \frac{\gamma(1-\gamma)\sigma^2}{2}$, and $\sigma_z := \gamma\sigma - \theta$.

3.2. Solution to the Free Boundary Problem

In this section, we derive the explicit solution to the free boundary problem (10). Before it, we introduce the following assumption to make the solution well-defined.

Assumption 1.

$$\beta_z > 0 \ \ and \ \ K := r_z + \frac{\beta_z - r_z}{\gamma} + \frac{(\gamma-1)}{2\gamma^2}\sigma_z^2 > 0.$$

Let n_1, n_2 be positive and negative roots of the following quadratic equation:

$$\frac{\theta^2}{2}n^2 + (\beta_z - r_z - \frac{\sigma^2}{2})n - \beta_z = 0. \tag{11}$$

We obtain the explicit form of $\mathcal{Q}(z)$ as in the following proposition. There are two cases: *(i)* case for a large ρ and *(ii)* case for a small ρ.

Proposition 1. *There exists $\rho^* > 0$ such that the following hold:*

(a) If $\rho > \rho^*$,

$$\mathcal{Q}(z) = \hat{C}_1 z^{n_1} + \hat{C}_2 z^{n_2} + \frac{1}{K}\frac{\gamma}{1-\gamma}z^{-\frac{1-\gamma}{\gamma}} + \frac{\tilde{\omega}}{\beta_z}\frac{1}{1-\gamma} + \frac{\phi}{r_z}z, \qquad z \in (\hat{z}_H, \hat{z}_L). \tag{12}$$

(b) If $0 < \rho < \rho^*$,

$$\mathcal{Q}(z) = C_1 z^{n_1} + C_2 z^{n_2} + \frac{1}{K}\frac{\gamma}{1-\gamma}z^{-\frac{1-\gamma}{\gamma}} + \frac{\tilde{\omega}}{\beta_z}\frac{1}{1-\gamma} + \frac{\phi}{r_z}z, \qquad z \in (z_H, z_L). \tag{13}$$

Here, the coefficients C_1, C_2, $\hat{C}_1$, $\hat{C}_2$ and the free boundaries z_H, z_L $\hat{z}_H$, $\hat{z}_L$ are given in the proof.

It is important to note that the size of ρ determines the boundary conditions that dictate the agent's choice of liquid and staked token holdings. Proposition 1 (a) and (b) provide the solution forms for each case. To accurately describe the optimal policy, we introduce the dual process $\mathcal{Z}_t$ and outline its behavior in each scenario, as summarized in Corollary 1:

$$\mathcal{Z}_t := \frac{1}{\omega}Y_t(k_t S_t)^\gamma = \frac{y}{\omega}e^{\beta t}H_t D_t (k_t S_t)^\gamma. \tag{14}$$

Here $\mathcal{Z}_t$ in (14) has a negative relationship with the wealth process. This is the reason why we set $z_H < z_L$ and $\hat{z}_H < \hat{z}_L$ since the lower boundaries z_H and $\hat{z}_H$ imply high wealth levels, and the upper boundaries z_L and $\hat{z}_L$ imply zero wealth.

Corollary 1. *The optimal policy (D_t^*, k_t^*) is given as follows.*

(a) Suppose $\rho > \rho^*$. Then, $d\mathcal{G}_t^- = 0$ for all $t \geq 0$. Here, D_t and $\mathcal{G}_t^+$ are adjusted so that $\hat{z}_H \leq \mathcal{Z}_t \leq \hat{z}_L$ for all $t \geq 0$. $dk_t^* = -\delta k_t^* dt$ during the times when $\hat{z}_H < \mathcal{Z}_t < \hat{z}_L$. $\mathcal{Z}_t$ never hits $\hat{z}_L$ and k_t^* increases whenever $\mathcal{Z}_t$ hits $\hat{z}_H$.

(b) *Suppose* $0 < \rho < \rho^*$. *Then,* $\mathcal{G}_t^+$ *and* $\mathcal{G}_t^-$ *are adjusted so that* $z_H \le \mathcal{Z}_t \le z_L$ *for all* $t \ge 0$. $D_t^* = 1$ *for all* $t > 0$. $dk_t^* = -\delta k_t^* dt$ *during the times when* $z_H < \mathcal{Z}_t < z_L$. k_t^* *increases (decreases) whenever* $\mathcal{Z}_t$ *hits* z_H (z_L).

When ρ is large, that is, when the cost of unlocking the staked tokens is substantially high, it is never optimal to unlock the staked tokens ($d\mathcal{G}_t^- = 0$ for all $t \ge 0$). What if the token price drops sufficiently so wealth becomes closer to zero? In this case, the staked position is completely hedged by shorting the liquid tokens ($x_t < 0$ so that $\pi_t = x_t^* + m_t^* = 0$) to ensure $W_t \ge 0$. This dynamic adjustment is controlled by optimal D_t^*. On the other hand, if the ratio of wealth to the holdings of staked tokens, $W_t/k_t^* S_t$, increases sufficiently, the agent stakes more tokens: the additional amount of newly staked tokens is $S_t d\mathcal{G}_t^+$ whenever $\mathcal{Z}_t$ hits $\hat{z}_H$.

Now we consider the case where ρ is small. In this case, the agent unlocks staked tokens whenever $\mathcal{Z}_t$ hits z_L (or the wealth process hits the zero boundaries) and increases staking whenever $\mathcal{Z}_t$ hits z_H (or $W_t/k_t^* S_t$ becomes sufficiently high). In this case, the total holdings of tokens π_t are positive when $W_t = 0$, unlike the case with high ρ and the agent unlocks the staked tokens in order to make $W_t \ge 0$.

3.3. Optimal Strategy

We provide the optimal strategy for the primal problem (3). To do so, we first establish the following duality relationship between the value function $V(w_0, s, k)$ defined in (3) and the dual value function $J(y, s, k)$ in (8).

Theorem 1 (Duality)**.**

$$V(w_0, s, k) = \inf_{y>0} \left(J(y, s, k) + y w_0 \right). \tag{15}$$

From the duality relationship (15), we obtain the following condition for optimality: for given $w_0 \ge 0$, $k > 0$, and $s > 0$, there exists a unique $y^* > 0$ such that $\frac{y^*(ks)^\gamma}{\omega} \in (0, z_L)$ and

$$w_0 = -ks\mathcal{Q}'\left(\frac{y^*(ks)^\gamma}{\omega}\right). \tag{16}$$

Proposition 2. *The optimal strategy* (c_t^*, x_t^*, k_t^*) *is given by*

$$c_t^* = m_t (\mathcal{Z}_t^*)^{-\frac{1}{\gamma}}, \tag{17}$$

$$x_t^* S_t = m_t \left(\frac{\theta - \gamma\sigma}{\sigma} \mathcal{Z}_t^* \mathcal{Q}''(\mathcal{Z}_t^*) - \mathcal{Q}'(\mathcal{Z}_t^*) \right) - m_t, \tag{18}$$

where m_t *is the total value of the staked token* $m_t := k_t^* S_t$, $\mathcal{Z}_t$ *is defined by* (14), *and the dynamics of* k_t^* *are described in Corollary* 1. *Then, the wealth process* $W_t^{c^*,x^*,k^*}$ *corresponding to the optimal strategy* (c_t^*, x_t^*, k_t^*) *is given by*

$$W_t^{c^*,x^*,k^*} = -m_t \mathcal{Q}'(\mathcal{Z}_t^*). \tag{19}$$

Note that there is the one-to-one correspondence between the dual variable y and wealth W_t by as $\mathcal{Q}'(z)$ is strictly decreasing in $z \in (z_H, z_L)$. From Theorem 1 and Proposition 2, we can derive the share of the optimal staking boundary. More precisely, for the current staking level k and the price s (thus the total value of the staked tokens is $m = ks$), the optimal wealth boundary $W_H(m)$ at which to increase staking and the ratio of maximum wealth to total staking $\mathcal{W}_m$ are defined respectively, as follows:

$$W_H(m) := -m\mathcal{Q}'(z_H) \qquad \text{or} \qquad \mathcal{W}_H := \frac{W_H(m)}{m} = -\mathcal{Q}'(z_H). \tag{20}$$

We can rewrite the optimal strategies by using the closed-form solution for $\mathcal{Q}(Z_t)$.

Corollary 2. *The optimal strategy* (c_t^*, x_t^*, k_t^*) *is rewritten as*

$$\begin{aligned} c_t^* &= K\left(W_t^{c^*,x^*,k^*} + \frac{\phi}{r_z}m_t\right) + Km_t\left(n_1C_1(\mathcal{Z}_t^*)^{n_1-1} + n_2C_2(\mathcal{Z}_t^*)^{n_2-1}\right) \\ x_t^*S_t + m_t &= \underbrace{\frac{\theta}{\gamma\sigma}\left(W_t^{c^*,x^*,k^*} + \frac{\phi m_t}{r_z}\right)}_{(i)} - \underbrace{\frac{\phi m_t}{r_z}}_{(ii)} \\ &\quad + \underbrace{\frac{\theta-\gamma\sigma}{\gamma\sigma}\left((1-\gamma+\gamma n_1)n_1C_1(\mathcal{Z}_t^*)^{n_1-1} + (1-\gamma+\gamma n_2)C_2(\mathcal{Z}_t^*)^{n_2-1}\right)}_{(iii)}, \end{aligned} \tag{21}$$

where m_t, $\mathcal{Z}_t^*$, *and* k^* *are same in Proposition 2.*

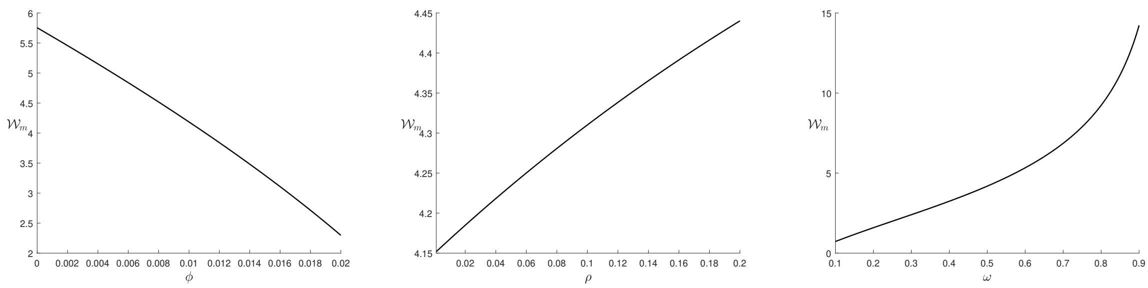
Corollary 2 clarifies the impact of the staking mechanism on the optimal consumption and portfolio decisions by identifying the similarity and thus eventually the difference between standard consumption and portfolio choice problems and the problem with staking. First, regarding the similar feature, given that ϕm_t is instantaneous income and thus $W_t + \phi m_t / r_z$ is interpreted as human wealth, note that the agent's total token holdings (the left-hand side of (21)) consist of three components in the following order: (i) the myopic demand, (ii) the hedging demand against stochastic income ϕm_t, and (iii) the liquidity hedging demand against the borrowing constraints. This decomposition is exactly the same as that resulted from the standard consumption and portfolio selection problem with stochastic income and the borrowing constraints (e.g., Ahn et al. (2019)) as ϕm_t is income from staking.

Second, regarding the specific feature of our case, it is important to note that given that the total token holdings ($x_tS_t + m_t$) consist of the three components, the cryptocurrency investor needs to decide how to allocate between the liquidity token holdings (x_tS_t) and the staking position (m_t). The optimal allocation rule between the two positions is specified by the inaction interval (z_H, z_L) and the evolution dynamics of k_t and D_t described in Proposition 1 and Corollary 1, respectively.

4. Implication

In this section, we investigate the implications of the model. First, we study the impact of the key parameters on the inaction interval (Section 4.1). Then, we investigate the optimal policies (Section 4.2), risk-taking dynamics (Section 4.3), and the impact of the cost and the reward on staking ratios (Section 4.4).

Before delving into the detailed explanation of the results, it is essential to acknowledge that the parameter values used in the figures and tables of this section have been chosen with a conservative approach. It should be noted that cryptocurrency returns and their volatility typically exhibit much higher values than those presented here. For instance, taking the example of bitcoin's return and volatility between September 3, 2013, and March 30, 2023, they amount to 38.5% and 61.52%, respectively (source: Choi et al. (2023b)). It is important to recognize that these numbers can significantly vary based on the selected sample period. Similar fluctuations can be observed for other tokens employing a staking mechanism. In this section, we cautiously demonstrate some quantitative implications by considering conservative estimates for the returns and return volatility.



(a) $\mathcal{W}_H$ as a function of ϕ (b) $\mathcal{W}_H$ as a function of ρ (c) $\mathcal{W}_H$ as a function of ω

Figure 1: The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \phi = 0.01, \rho = 0.02, \omega = 0.5$

4.1. Inaction Interval

What is the frequency at which the agent adjusts their staking position? Proposition 3 offers comparative static results regarding the size of the inaction region in relation to ϕ, ρ, and ω. These results shed light on the impact of these parameters on how the agent manages the staking position.

Proposition 3. *The following hold:*

(a) The inaction interval decreases with ϕ: $\frac{\partial \mathcal{W}_H}{\partial \phi} < 0$.

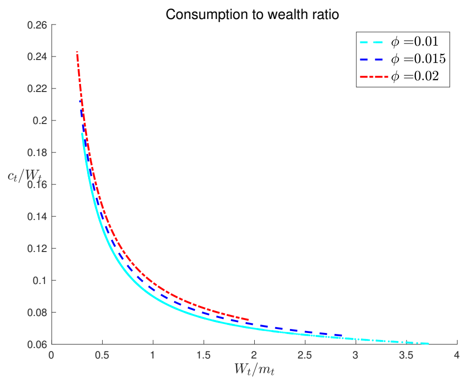
(b) If $0 < \rho < \rho^*$, the inaction interval increases with ρ: $\frac{\partial \mathcal{W}_H}{\partial \rho} > 0$.

(c) The inaction region increases with ω: $\frac{\partial \mathcal{W}_H}{\partial \omega} > 0$.

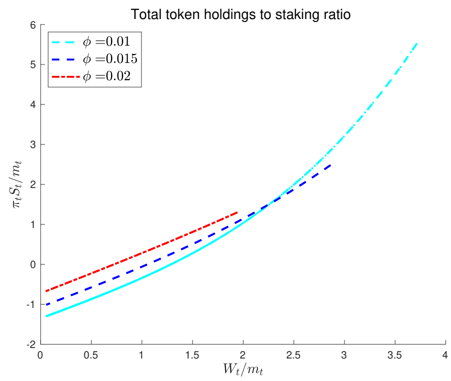
Based on Proposition 3, we can infer that the agent will adjust their staking position more frequently under the following conditions: (a) as ϕ increases, (b) as ρ decreases, or (c) as ω decreases. Note regarding the condition of Proposition (b) 3 that if $\rho > \rho^*$, the inaction interval is independent of ρ as the agent never liquidates the staking position.

Figure 1 illustrates the behavior of $\mathcal{W}_H$ with respect to these parameters: panel (a) shows the relationship with ϕ, panel (b) with ρ, and panel (c) with ω. The results regarding (a) and (b) come out as easily expected. More precisely, while a higher staking reward encourages greater token staking (Proposition 3(a)), a lower cost leads to more frequent adjustments in the staking position (Proposition 3(b)).

What about the trading frequency of active DAO participants? The agent will adjust their staking position more frequently as ω decreases. Figure 1 (c) illustrates the behavior of $\mathcal{W}_H$ with respect to ω. Interestingly, this finding reveals that active participants change their staked position more frequently than general investors, which may seem counter-intuitive at first. One might expect active DAO participants to be long-term stakers. However, active participants derive higher utility gains from staking, which motivates them to increase the number of staked tokens even in response to small increases in wealth. Therefore, the trading frequency of active participants is higher than that of general agents. We also empirically verify this prediction (see Appendix A).

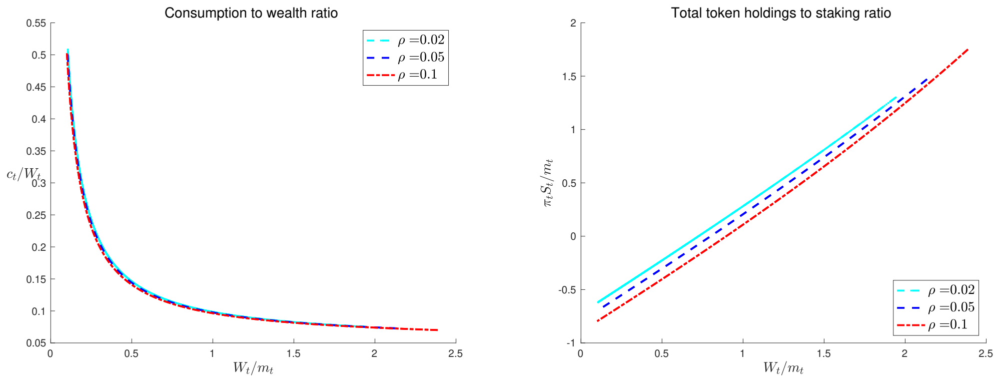


(a) $\frac{c_t}{W_t}$



(b) $\frac{\pi_t}{k_t}$

Figure 2: Consumption and liquid token holdings in the inaction interval with different ϕ's. The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \rho = 0.02, \delta = 0.03, \omega = 0.5$



(a) $\frac{c_t}{W_t}$ (b) $\frac{\pi_t}{k_t}$

Figure 3: Consumption and liquid token holdings in the inaction interval with different ρ's. The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \phi = 0.02, \delta = 0.03, \omega = 0.5$

4.2. Optimal Consumption and Risk-taking

Figures 2 and 3 provide visual representations of the optimal policies in our model. Specifically, Figure 2(a) displays the optimal consumption-to-wealth ratio, $\frac{c_t^*}{W_t}$, and Figure 2(b) illustrates the ratio of total token holdings to staked token holdings, $\frac{\pi_t S_t}{m_t} = \frac{\pi_t}{k_t}$, within the inaction interval.

From Figures 2(a) and 3(a), we observe that the ratio of consumption to wealth decreases as wealth increases, while the absolute amount of consumption increases with wealth. This finding can be directly inferred from equations (17) and (28). In contrast, as wealth increases, the liquid token holdings and thus the total token holdings increase (see Figures 2(b) and 3(b)), which results from the fact that the liquidity hedging demand against the borrowing constraints is greatly relaxed (Corollary 2). This implies that as the agent's wealth grows, they allocate more funds towards liquid token holdings rather than increasing consumption. As a result, the ratio of consumption to wealth decreases in this scenario. In other words, when the token price is increasing (decreasing), the agent chooses to take more (less) risk.

4.3. Risk-taking Dynamics

Based on the result in Section 4.2, we conduct further investigation into the quantitative impact of liquidity demand over time. Prior to examining the time-series analysis on risky token investment, it is worth noting the significant fluctuations in the ratio of liquidity token holdings to staked tokens, represented as $\frac{x_t}{k_t} = \frac{\pi_t}{k_t} - 1$, within the inaction interval. For instance, in Figure 2(b), when $\phi = 0.01$, $\frac{\pi_t}{k_t}$ varies widely, ranging from approximately -2 to 5. This example suggests that the agent's token holdings can differ substantially between good and bad times in the cryptocurrency market.

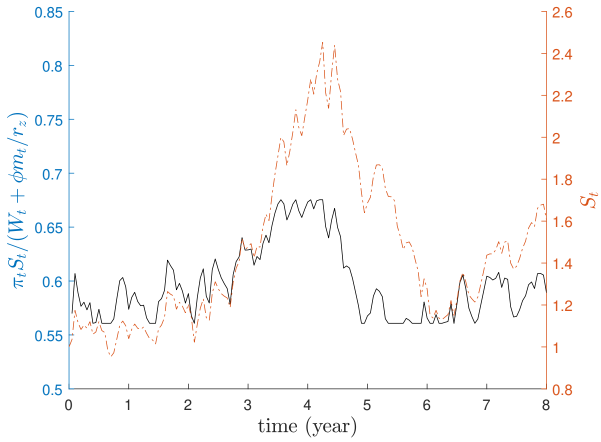


Figure 4: Sample path of the cryptocurrency price S_t (red dotted lines) and the corresponding time series of $\frac{\pi_t S_t}{W_t+\phi m_t/r_z}$ (black solid lines). The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.15, \sigma = 0.2, r = 0.03, \phi = 0.02, \delta = 0.03, \omega = 0.2$, and $\rho = 0.05$.

With the above observation in mind, let us consider the dynamics of the token holdings. Figure 4 displays a sample path of the cryptocurrency price S_t along with the corresponding time series of the ratio of the total token investment to human wealth, expressed as $\frac{\pi_t S_t}{W_t+\phi m_t/r_z}$. Recalling the context of the standard Merton model, this ratio is constant and independent of the price only if the agent invests in a token without a staking mechanism and derives no utility from holding the token: $\omega = 1$ and $\phi = 0$. However, other than this special case, there is a strong positive correlation between the ratio and the cryptocurrency price in our model. For example, as shown in Figure 4, this ratio is 22.3% higher (reaching 0.685) during favorable periods compared to unfavorable periods (as low as 0.560).

The strong positive correlation highlights the role of the staking mechanism. To make it easier to understand, let us consider the scenario where the price is increasing (decreasing). In this case, both wealth and the ratio experience an increase (decrease). Consequently, the absolute difference in token investments between favorable and unfavorable times is significantly higher than that between the maximum and minimum ratios. For instance, in Figure 4 the total token investment $\pi_t S_t = 1.4668$ at $t = 4$ exceeds $\pi_t S_t = 0.7223$ at $t = 6$ by over 100%, while the ratio at $t = 4$ is only 22.3% higher that the ratio at $t = 6$. This dynamic behavior is fairly relevant to stakers' investment patterns. To elaborate further, let's consider two distinct investors of the cryptocurrency with a staking mechanism, one being a staker and the other a non-staker. Non-staking is not considered optimal due to the presence of a staking

		max	min	(max-min)/min (%)
	0.010	0.8418	0.7618	10.49
	0.015	0.6311	0.5033	25.39
ϕ	0.020	0.4966	0.3775	31.56
	0.025	0.4060	0.3039	33.62
	0.030	0.3422	0.2559	33.70
	0.05	0.4966	0.3775	31.56
	0.06	0.5128	0.3861	32.83
ρ	0.07	0.5273	0.3942	33.76
	0.08	0.5403	0.4019	34.43
	0.09	0.5521	0.4093	34.88
	0.10	0.4966	0.3775	31.56
	0.15	0.5945	0.4736	25.54
ω	0.20	0.6754	0.5608	20.43
	0.25	0.7451	0.6490	14.81
	0.30	0.8022	0.7446	7.72

Table 1: The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.15, \sigma = 0.2, r = 0.03, \phi = 0.02, \delta = 0.03, \omega = 0.1$, and $\rho = 0.05$.

reward. In our context, a non-staker refers to an investor who cannot stake the tokens for any reason. For instance, this investor exclusively trades the cryptocurrency on a centralized exchange that does not offer staking services for the token. In this case, our result implies that the staker's token holdings are notably higher in good times and relatively lower during bad times compared to the non-staker's holdings.

Table 1 shows the maximum and minimum values of the ratio of the total token investment to human wealth and the percentage difference between the maximum and minimum ratios for each value of (a) ω, (b) ϕ, and (c) ρ. As shown in the table, it is common that the ratio fluctuates more than 20-30%. The maximum and minimum ratios are generally monotonic in ω, ϕ, and ρ, while their percentage difference is not monotonic. In summary, Table 1 demonstrates that the reasonable values of (ω, ϕ, ρ) can generate similar or even larger fluctuations in the investment ratio as in Figure 4.

4.4. Effects of Cost and Staking Reward

First, Figure 5 provides insights into the impact of ρ on the staking ratio when the wealth-to-staking ratio is held constant. Irrespective of the level of the wealth-to-staking ratio and the

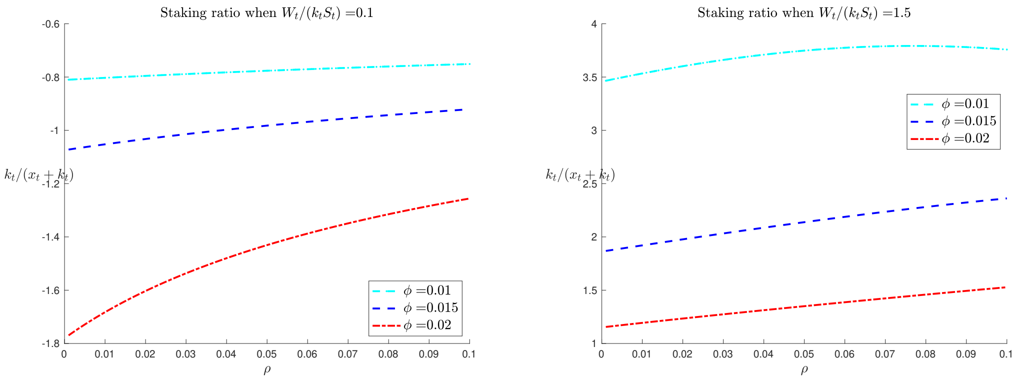


Figure 5: Effects of the lockup cost (ρ) on staking ratio ($k_t/(x_t + x_t)$). The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \delta = 0.03, \omega = 0.5$.

staking reward, it is observed that the staking ratio consistently increases with the locking cost. This implies that as the cost of unlocking staked tokens increases, the agent tends to allocate a larger portion of their tokens to staking.

Next, we analyze the impact of the staking reward ϕ on the consumption to wealth ratio. Recall that staking more tokens or unlocking staked tokens only occurs when the wealth-to-staking ratio, W_t/m_t, reaches the upper or lower boundary.

Figure 6 depicts the effects of the staking reward on the consumption to wealth ratio, considering two scenarios: fixing $W_t/m_t = 0.1$ in the left panel and $W_t/m_t = 1.5$ in the right panel. In the left panel, the proportion of staked tokens relative to the total token holdings is higher compared to the right panel.

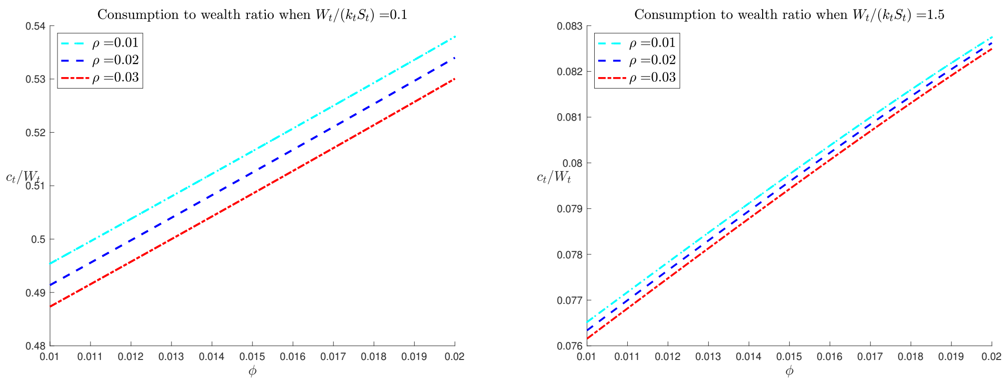
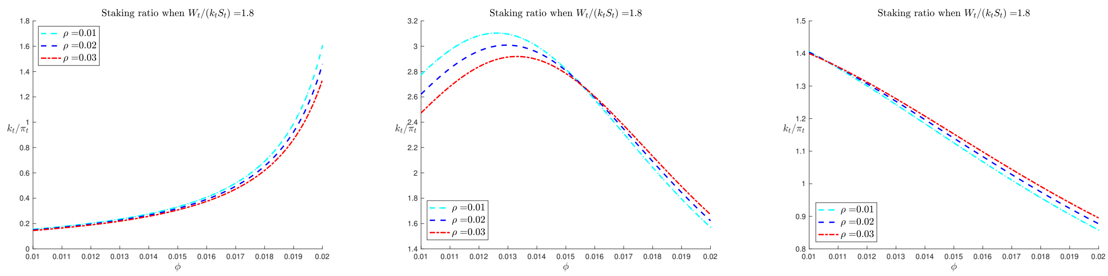


Figure 6: Effect of staking reward (ϕ) on consumption to wealth ratio. The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \delta = 0.03, \omega = 0.5$

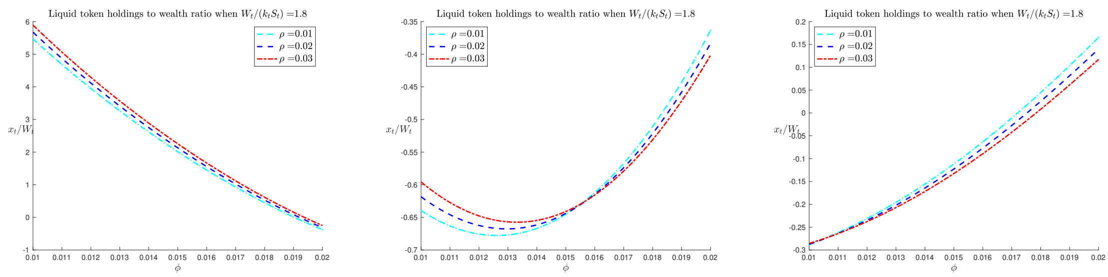
The impact of the staking reward on the consumption to wealth ratio is twofold. First, the consumption to wealth ratio increases with a higher staking reward. Second, when the proportion of staked tokens is higher, the change in the consumption to wealth ratio becomes more sensitive to variations in the staking reward. This can be observed by comparing the slopes of the graphs in the left and right panels of Figure 6. The larger scale of the y-axis in the right panel indicates a more pronounced increase in consumption as the staking reward ϕ rises.

On the other hand, the cost ρ has a negative effect on consumption. As the cost of unlocking staked tokens increases, the consumption level decreases. This is evident when comparing the graphs with different values of ρ in Figure 6 ($\rho = 0.01$, 0.02, and 0.03, respectively). The reason behind this result is that a higher cost reduces the net return from investments, leading to a decrease in consumption.



(a) ω=0.9 (b) ω=0.65 (c) ω=0.5

Figure 7: Effect of ϕ on the staking ratio, $k_t/(x_t + k_t)$, when $W_t/m_t = 1.8$ (fixed). The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \delta = 0.03$

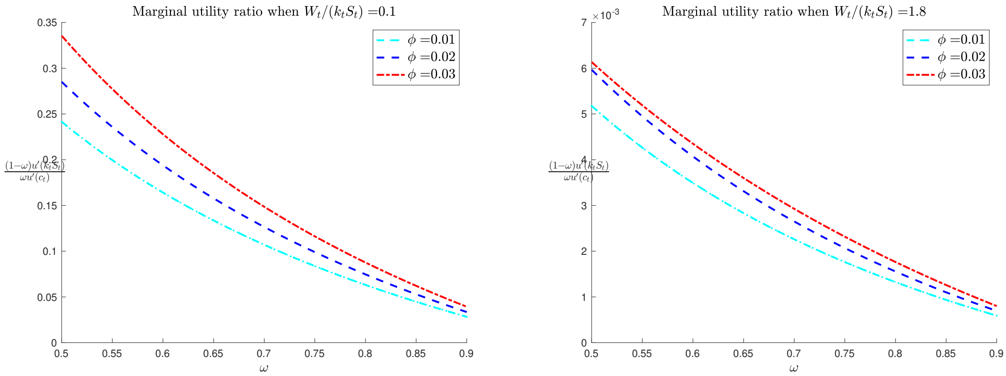


(a) ω=0.9 (b) ω=0.65 (c) ω=0.5

Figure 8: Effect of ϕ on liquid token to wealth ratio, x_tS_t/W_t, when $W_t/m_t = 1.8$ (fixed). The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \delta = 0.03$

What about the impact of the staking reward on the staking ratio, $\frac{k_t}{k_t+x_t}$? One might initially expect that the staking ratio increases as the reward increases. However, this is only true for general investors who do not derive significant utility from staking tokens. As shown in Figure

7(a) with $\omega = 0.9$, the agent increases the staking ratio to maximize the staking reward as the reward ϕ increases. However, for active investors with small ω who gain substantial utility from staking tokens, the relationship is reversed. As depicted in Figure 7(c), the staking ratio actually decreases with increasing staking reward.



(a) $\frac{W_t}{k_t S_t} = 0.1$ (b) $\frac{W_t}{k_t S_t} = 1.8$

Figure 9: The ratio of marginal utilities from consumption and staking. The parameter set is given by $\gamma = 2, \beta = 0.07, \mu = 0.08, \sigma = 0.25, r = 0.03, \rho = 0.02, \delta = 0.03$

To understand the above result, it is important to recognize that the staking reward represents an additional income stream, effectively increasing the total wealth of the agent. As a result of this income effect, the agent optimally increases their total token holdings, encompassing both x_t and k_t, as the staking reward ϕ increases. Therefore, the reason why the staking ratio increases (decreases) with ϕ when ω is small (large) is due to the differential impact on the liquid token holdings and stake token holdings. Specifically, when ω is small, the increase in liquid token holdings surpasses the increase in stake token holdings as ϕ increases. In contrast, when ω is large, the increase in stake token holdings outweighs the increase in liquid token holdings as ϕ increases.

With the above income effect in mind, we can analyze the marginal rate of substitution (MRS) between staking and consumption. The MRS represents the ratio of the marginal utility from staking to the marginal utility from consumption:

$$MRS_{kc} := \frac{(1-\omega)u'(kS_t)}{\omega u'(c_t)}.$$

As observed in Figure 9 for different categories of token holders, the marginal rate of substitution (MRS_{kc}) is higher for active participants of the DAO compared to general investors: MRS_{kc} decreases as ω decreases. This indicates that token holders with smaller ω experience a relatively larger utility loss when unlocking staked tokens in the event of adverse shocks.

As risk-averse agents, they respond by reducing their staking ratio while increasing their total token holdings due to the income effect.

5. Extension: Case with an Additional Risky Asset

Previously, stock markets and cryptocurrency markets were regarded as separate entities in the sense that the majority of stock investors did not participate in cryptocurrency markets and vice versa. However, with the increasing popularity of cryptocurrency investing, there has been a growing trend of investors engaging in both markets. To account for this evolving landscape, we extend our model in this section to incorporate an additional risky asset, such as stocks.

In this section, we first set up an extended model. After presenting the solution analysis, we provide new implications from the extended model. Again it is important to note that a key distinction between the two markets is the absence of staking in the stock market. Therefore, the additional risky asset can encompass cryptocurrencies like Bitcoin, Dogecoin, and others, as they lack a staking mechanism.

5.1. Model and Solution

Assume that P_t represents the (cum-dividend) price of the stock or the price of the non-staking cryptocurrency token:

$$dP_t/P_t = \mu_p dt + \sigma_p(\varrho dB_t + \sqrt{1-\varrho^2}d\Xi_t),$$

where Ξ_t is another standard Brownian motion which is independent of B_t, and $\varrho \in (-1,1)$ is the correlation between the mean returns of the staking asset S_t and non-staking asset P_t. Then, the wealth dynamics in (2) are rewritten as

$$\begin{aligned} dW_t =& [rW_t + (\mu - r)\pi_t S_t + (\mu_p - \mu)\pi_{p,t} + \phi k_t S_t - c_t]dt \\ & -S_t d\mathcal{G}_t^+ + (1-\rho)S_t d\mathcal{G}_t^- + \sigma\pi_t S_t dB_t + \sigma_p \pi_{p,t} P_t\left(\varrho dB_t + \sqrt{1-\varrho^2}d\Xi_t\right). \end{aligned} \tag{22}$$

In order to avoid introducing new notations and to precisely describe the optimal strategy of the extended model, we need to redefine the coefficients r_z, β_z, and σ_z as follows:

$$r_z = \delta,\ \beta_z = \beta - (1-\gamma)(\mu - \delta) + \frac{\gamma(1-\gamma)\sigma^2}{2},\ \text{and}\ \sigma_z = \sqrt{(\gamma\sigma - \theta)^2 + \theta_p^2}, \tag{23}$$

where $\theta_p := \dfrac{-\sigma_p\varrho(\mu - r) + \sigma(\mu_p - r)}{\sigma\sigma_p\sqrt{1-\varrho^2}}$. In addition, $\mathcal{Z}_t$ is redefined as

$$\mathcal{Z}_t = \frac{y}{\omega}e^{(\beta - r - \frac{1}{2}(\theta + \theta_p^2))dt - \theta B_t - \theta_p \Xi_t} D_t(k_t S_t)^\gamma. \tag{24}$$

Likewise, all the other parameters (e.g., $z_H, z_L, \hat{z}_H, \hat{z}_L$) are also redefined in accordance with (23). Then, we have the following proposition valid under these redefined coefficients and $\mathcal{Z}_t$-process.

Proposition 4. *The optimal strategy* $(c_t^*, x_t^*, k_t^*, \pi_{p,t}^*)$ *is given by*

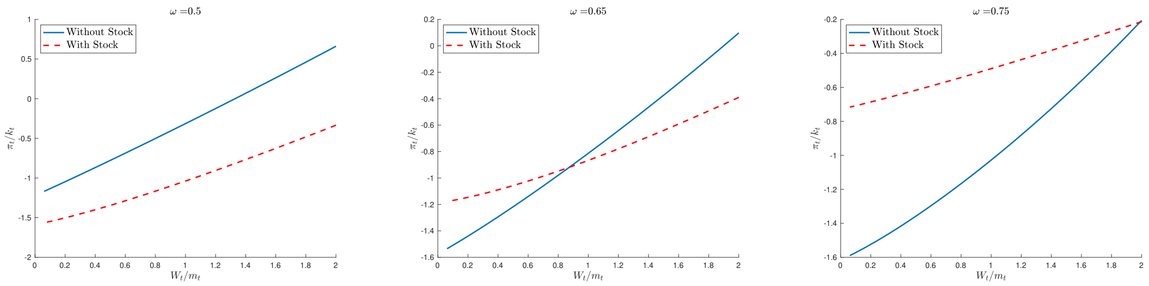
$$c_t^* = m_t(\mathcal{Z}_t^*)^{-\frac{1}{\gamma}}, \tag{25}$$

$$x_t^* S_t = m_t \left(\frac{\theta\sigma_p\sqrt{1-\varrho^2} - \theta_p\sigma_p\varrho - \gamma\sigma\sigma_p\sqrt{1-\varrho^2}}{\sigma\sigma_p\sqrt{1-\varrho^2}} \mathcal{Z}_t^* \mathcal{Q}''(\mathcal{Z}_t^*) - \mathcal{Q}'(\mathcal{Z}_t^*) \right) - m_t, \tag{26}$$

$$\pi_{p,t}^* P_t = m_t \frac{\theta_p}{\sigma_p\sqrt{1-\varrho^2}} \mathcal{Z}_t^* \mathcal{Q}''(\mathcal{Z}_t^*) \tag{27}$$

where $m_t := k_t^* S_t$ *and* $\mathcal{Z}_t^*$ *is given by (24). The dynamics of* k_t^* *are the same as what is described in Corollary 1 with redefined coefficients (23) and* $\mathcal{Z}_t^*$*. Then, the wealth process* $W_t^{c^*, x_t^*, k_t^*, \pi_{p,t}^*}$ *corresponding to the optimal strategy* $(c^*, x^*, k_t^*, \pi_{p,t}^*)$ *is given by*

$$W_t^{c^*, x^*, k^*, \pi_{p,t}^*} = -m_t \mathcal{Q}'(\mathcal{Z}_t^*). \tag{28}$$



(a) $\omega = 0.5$ (b) $\omega = 0.65$ (c) $\omega = 0.75$

Figure 10: $\frac{\pi_t}{k_t}$ with the stock (red dotted line) versus without the stock (blue line). The parameters are given by $\gamma = 3, \beta = 0.07, \mu = 0.15, \mu_p = 0.07, \sigma = 0.4, \sigma_p = 0.2, r = 0.02, \phi = 0.025, \rho = 0.1$, and $\varrho = 0$.

5.2. Implications

Here we discuss the new implications of the extended model based on Proposition 4.

Optimal Staking Ratios: What is the difference in optimal staking ratios between when the agent participates in both markets and when the agent only participates in the cryptocurrency market? Figure 10 illustrates the comparison of π_t/k_t (the inverse of the staking ratio) in both scenarios. It is important to note that when an investor participates in both markets, both the staking amount (k_t) and the total cryptocurrency investment ($\pi_t = k_t + x_t$) decrease compared to when the investor solely participates in the cryptocurrency market. This result is in accordance with the standard diversification argument.

Figure 10(a) demonstrates that, in this case, the investor reduces their total cryptocurrency holdings more than the staking amount when ω is small (active participants), resulting in an increase in the staking ratio k_t/π_t (equivalently decrease in π_t/k_t in the figure). Figure 10(c) illustrates the situation in which the staking ratio decreases when ω is large (general agents). If ω is intermediate, there is a crossover depending on financial wealth (Figure 10(b)). In this case, the staking ratio increases (decreases) when wealth is large (small). That is, the wealth effect dominates the liquidity hedge need as wealth increases for the intermediate level of ω.

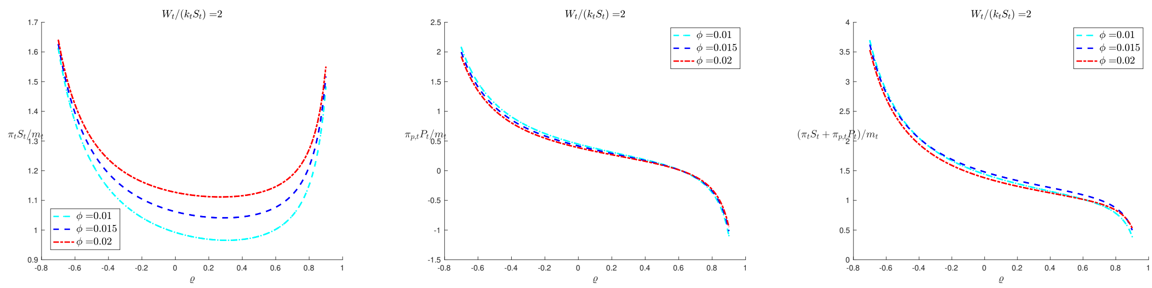
Case with the Additional Asset being a Stock: Next, we explore the effects of the correlation (ϱ) between the stock market and the cryptocurrency market on asset holding positions. It is important to note that as long as there is no perfect correlation between the two markets, the agent allocates a substantial (nontrivial) amount to the staking position. Thus, we examine the liquid cryptocurrency position and the stock position, both normalized by the staking amount. Specifically, we consider the following three ratios:

(a) Crypto holdings/Staking (i.e., $\frac{\pi_t S_t}{m_t}$)

(b) Stock holdings/Staking (i.e., $\frac{\pi_{p,t} S_t}{m_t}$)

(c) Total asset holdings/Staking (i.e., $\frac{\pi_t S_t + \pi_{p,t} P_t}{m_t}$).

These ratios provide insights into the relative proportions of cryptocurrency holdings, stock holdings, and total asset holdings compared to the staking amount.



(a) $\frac{\text{Crypto holdings}}{\text{Staking}}$ (b) $\frac{\text{Stock holdings}}{\text{Staking}}$ (c) $\frac{\text{Total asset holdings}}{\text{Staking}}$

Figure 11: Effect of ϱ when $W_t/m_t = 2.0$ (fixed): the three ratios as a function of ϱ. The other parameters are given by $\gamma = 2, \beta = 0.07, \mu = 0.15, \mu_p = 0.7, \sigma = 0.4, \sigma_p = 0.2, r = 0.03, \delta = 0.03$

Figure 11 depicts the three aforementioned ratios with respect to the correlation (ϱ) between the two markets. Ratio (a) exhibits a U-shaped pattern in relation to ϱ, while ratio (b) generally decreases as ϱ increases. This implies that when the two markets demonstrate negative correlations, both crypto and stock holdings decline with increasing ϱ. On the other hand, when the two markets display positive correlations, crypto holdings increase, while stock holdings decrease with ϱ. In this case, as ϱ increases and becomes positive, the decrease in stock holdings outweighs the increase in crypto holdings. Consequently, as observed in Figure 11(c), the total asset holdings decrease with ϱ.

To summarize, Given that the agent has significant crypto holdings in staking, the total asset holdings (i.e., liquid crypto holdings and stock holdings) diminish as the correlation between the two markets increases. This result is somewhat intuitive given that the Sharpe ratio of the cryptocurrency market is higher than that of the stock market (while the former is dramatically changing over time).

In Figure 11, we take the return and volatility of the stock as 7% and 20%, respectively, which are representative values for a typical U.S. stock market, such as the S&P 500 or Nasdaq. Moreover, to highlight the typical difference between the two markets, we assume that the return and volatility of the cryptocurrency asset are approximately twice as high as those of the stock. It is worth noting that common cryptocurrencies like Bitcoin and Ethereum often exhibit much higher returns and volatilities, although these values can vary significantly depending on the sample period. Importantly, the results discussed in the figure are further strengthened when we employ higher return and volatility values for the cryptocurrency asset.

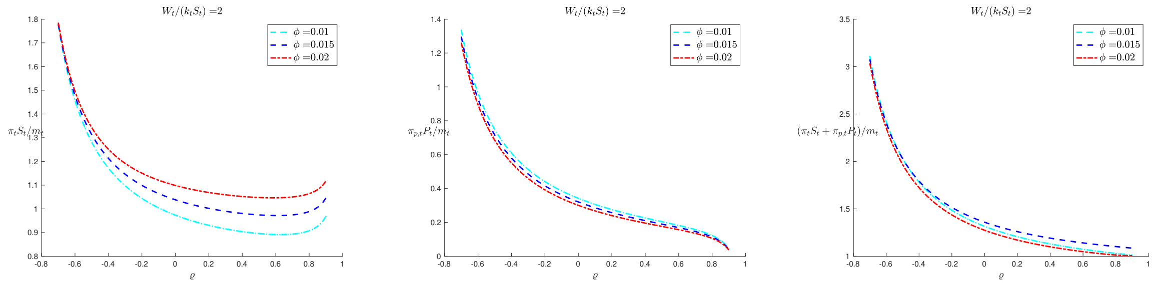
Case with the Additional Asset being a Non-staking Crypto: In the case where the agent solely invests in the crypto market, which consists of two different tokens–one being a staking cryptocurrency and the other a non-staking cryptocurrency–we can consider the correlation ϱ between the two. Similar to the previous case, we examine the liquid staking crypto position and the non-staking crypto position, both normalized by the amount of staking. Specifically, we consider the following ratios:

(a) Staking Crypto holdings/Staking (i.e., $\frac{\pi_t S_t}{m_t}$)

(b) Non-staking Crypto holdings/Staking (i.e., $\frac{\pi_{p,t} S_t}{m_t}$)

(c) Total asset holdings/Staking (i.e., $\frac{\pi_t S_t + \pi_{p,t} P_t}{m_t}$)

The implications in this case closely resemble those in the scenario where the additional risky asset is a stock, as shown in Figure 12. The main difference only comes from the return structure of the additional risky asset. In Figure 12, we assume that the return and volatility of



(a) $\frac{\text{Staking Crypto holdings}}{\text{Staking}}$ (b) $\frac{\text{Non-staking Crypto holdings}}{\text{Staking}}$ (c) $\frac{\text{Total asset holdings}}{\text{Staking}}$

Figure 12: Effect of ϱ when $W_t/m_t = 2.0$ (fixed): the three ratios as a function of ϱ. The other parameters are given by $\gamma = 2, \beta = 0.07, \mu = 0.15, \mu_p = 0.15, \sigma = 0.4, \sigma_p = 0.4, r = 0.03, \delta = 0.03$

the additional asset are the same as those of the staking token, which is based on the observation that many cryptocurrencies exhibit a fairly similar risk-return profile. By assuming similar risk-return characteristics for both staking and non-staking cryptocurrencies, the findings from Figure 12 align with those from Figure 11. This similarity suggests that the correlations and their effects on the asset holdings remain consistent across different scenarios, regardless of whether the additional asset is a stock or a non-staking cryptocurrency.

With the given parameterization, a noteworthy finding in this case is that the non-staking cryptocurrency position becomes significantly smaller when the correlation ϱ is relatively high. This finding holds largely true for higher values of the return and volatility of the additional non-staking cryptocurrency. This robust result carries interesting implications. Considering that the returns of the majority of cryptocurrencies exhibit a strong correlation with Bitcoin, it suggests that an agent who stakes a particular token has a tendency to hold primarily the corresponding tokens rather than other non-staking cryptocurrencies. This observation highlights the importance of the correlation between different cryptocurrencies and its influence on an investor's asset allocation decisions.

6. Conclusion

In this paper, we have addressed the optimal staking decision problem faced by cryptocurrency investors. Our model captures the trade-off between the utility gained from staking tokens and the costs or frictions associated with unlocking them. Through the duality relationship, we derived a variational inequality problem with two singular controls. Our analysis revealed that the optimal staking policy involves an inaction interval for the wealth-to-staking ratio, where the investor refrains from changing their staking position. Additionally, we explored various implications of the staking reward and costs for optimal policies. We also perform simple empirical tests to show that active investors trade more frequently. Finally, we extend the model to the case in which investors participate in both the cryptocurrency market with staking and the stock market.

Our model abstracts from certain aspects of reality, such as time-varying returns or jump shocks in cryptocurrency prices. Investigating the effects of these factors on the liquid and staking positions of cryptocurrency investors would be an intriguing avenue for future research.

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Appendix

A. Simple Empirical Analysis

We conduct simple empirical tests to examine our theoretical predictions. However, it is important to note the following caveat before presenting the results: cryptocurrency investors often have multiple wallet addresses serving different purposes. Some of these wallets may be on-chain, while others may be held in centralized exchanges. Consequently, accurately identifying all the wallet addresses and thus the total portfolios held by a single agent poses a considerable challenge. Due to this issue, we include the empirical results in the appendix. Nonetheless, we proceed to carefully argue that the presented empirical findings align with our theoretical prediction.

A.1. Data Description

Table 2: Summary statistics

quantity is the token balance of an individual token holder (a wallet address) and *trx_n* is the total number of transactions of the token executed by the wallet address. We collect the on-chain data from https://etherscan.io. For each token, we choose the 1000 largest token wallet addresses and their transaction numbers from the genesis block time to June 21, 2023. We put the total transaction number as 10000 to reduce the effect of sample bias if the number is greater than 10000.

	mean	sd	min	median	max
AAVE					
quantity	15033.54	129864.60	327.67	975.69	3302672
trx_n	354.63	1535.83	1	5	10000
LIDO					
quantity	982277.50	5709589	15805.87	61716.97	1.10e+08
trx_n	219.30	1239.08	1	4	10000
Maker					
quantity	948.40	7026.36	20.23	74.97	190987.1
trx_n	326.24	1468.74	1	5	10000

We use the transaction data of three DAO Tokens (Aave, Lido Dao, and Maker) obtained from Etherscan (https://etherscan.io/). We collected the trading history of the top 1000 wallet addresses for each DAO's token holders from the genesis block date to June 20, 2023. It is important to note that the dollar value of the tokens held by wallet addresses below the top

1000 is at most 1000 USD, indicating that they are unlikely to be active DAO investors.

Table 2 presents the summary statistics for each coin. The *quantity* of each token represents the total number of tokens in each wallet address, while *trx_n* indicates the total number of transactions executed by each wallet address during the specified period. The first row displays the mean, standard deviation, minimum, median, and maximum values for *quantity*. Similarly, the second row provides the same information for *trx_n*. For instance, among the 1000 wallet addresses of Aave, the mean value for the total number of transactions is 15,033.54.

Table 3: Regression results

	(1) *AAVE* total	(2) *AAVE* w/o protocols	(3) *LIDO* total	(4) *LIDO* w/o protocols	(5) *Maker* total	(6) *Maker* w/o protocols
log(quantity)	345.03***	56.85***	109.70***	46.07***	198.88***	40.07***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Constant	-2180.37***	-314.72**	-1043.31***	-473.31***	-612.44***	-117.48**
	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.010)
Obs.	1000	927	1000	931	1000	922
R^2	0.11	0.02	0.02	0.02	0.04	0.02

p-values in parentheses
* $(p < 0.05)$,** $(p < 0.01)$,*** $(p < 0.001)$

A.2. Analysis

We utilize this dataset to test one of our key findings: the reduction of the inaction interval as ω increases. Based on this, we propose the following empirical prediction: individuals who are more actively involved in the DAO tend to engage in more frequent trading. We posit that a higher inclination to participate in the DAO correlates with larger holdings of governance tokens. Our regression model is

$$trx\_n = \alpha + \beta \times \log(quantity) + \epsilon, \tag{A.1}$$

where α and β are the regression coefficients and ϵ represents an additive error term. Note that although we use the logarithmic transformation for *trx_n*, using the absolute value instead does not significantly alter the results.

Table 3 presents the results of a simple regression analysis that examines the impact of token holdings on trading frequency. The odd columns display the results obtained using the entire sample, including wallet addresses from centralized exchanges and other Defi protocols. It should be noted that the wallet addresses of these protocols may or may not be regarded as active participants. To ensure the robustness of our findings, we conduct a separate regression analysis excluding these protocols, and the results are displayed in the even columns for each token. Notably, significant results were observed in both cases.

B. Proofs

B.1. Variational Inequality for J

By applying the dynamic programming principle, we derive the following variational inequality for J:

$$\begin{cases} \mathcal{L}_{y,s,k} J + \omega \tilde{u}\left(\frac{y}{\omega}\right) + (1-\omega)u(ks) + \phi ysk = 0 & \text{if } (1-\rho)ys < \partial_k J < ys \text{ and } \partial_y J < 0, \\ \mathcal{L}_{y,s,k} J + \omega \tilde{u}\left(\frac{y}{\omega}\right) + (1-\omega)u(ks) + \phi ysk \leq 0 & \text{if } \partial_k J = ys \text{ or } \partial_k J = (1-\rho)ys, \\ \mathcal{L}_{y,s,k} J + \omega \tilde{u}\left(\frac{y}{\omega}\right) + (1-\omega)u(ks) + \phi ysk \geq 0 & \text{if } \partial_y J = 0, \end{cases} \tag{B.1}$$

where $\mathcal{L}_{y,s,k}$ is given by

$$\mathcal{L}_{y,s,k} := \frac{\theta^2}{2} y^2 \partial_{yy} + \frac{\sigma^2}{2} s^2 \partial_{ss} - \theta\sigma\partial_{sy} + (\beta - r)y\partial_y + \mu s \partial_y - \delta k \partial_k - \beta.$$

Here, ∂_i and ∂_{ij} represent the first and second partial derivatives with respect to $i, j \in \{y, s, k\}$. Note that under transform (9), the conditions $\partial_k J = 0$ and $\partial_y J = 0$ are equivalent to $(1-\gamma)\mathcal{Q} + \gamma z \mathcal{Q}' = 0$ and $\mathcal{Q}' = 0$, respectively.

B.2. Proof of Proposition 1

We first consider the case $\omega = 1$ for simplicity. In this case, we need the assumption $\phi > r_z$.

In this case, we can rewrite the variational inequality (10) for $\mathcal{Q}$ as follows:

$$\begin{cases} \mathcal{L}_z \mathcal{Q} + \tilde{u}(z) + \phi z = 0 & \text{if } (1-\rho)z < (1-\gamma)\mathcal{Q} + \gamma z \mathcal{Q}' < z \text{ and } \mathcal{Q}' < 0, \\ \mathcal{L}_z \mathcal{Q} + \tilde{u}(z) + \phi z \leq 0 & \text{if } (1-\gamma)\mathcal{Q} + \gamma z \mathcal{Q}' = z \text{ or } (1-\gamma)\mathcal{Q} + \gamma z \mathcal{Q}' = (1-\rho)z, \\ \mathcal{L}_z \mathcal{Q} + \tilde{u}(z) + \phi z \geq 0 & \text{if } \mathcal{Q}' = 0. \end{cases} \tag{B.2}$$

We suppose that there exists the critical level $\rho^* > 0$ such that for $0 < \rho < \rho^*$, we can find a convex function $\mathcal{Q}(z)$ satisfying the following free boundary problem with z_L and z_H:

$$\begin{cases} \mathcal{L}_z \mathcal{Q} + \tilde{u}(z) + \phi z = 0 \text{ for } z_H < z < z_L, \\ (1-\gamma)\mathcal{Q}(z_H) + \gamma z_H \mathcal{Q}'(z_H) = z_H, \quad \mathcal{Q}'(z_H) + \gamma z_H \mathcal{Q}''(z_H) = 1, \\ (1-\gamma)\mathcal{Q}(z_L) + \gamma z_L \mathcal{Q}'(z_L) = (1-\rho)z_L, \quad \mathcal{Q}'(z_L) = 0. \end{cases} \tag{B.3}$$

For $z_H < z < z_L$, we can put a general solution of $\mathcal{Q}(z)$ by

$$\mathcal{Q}(z) = C_1 z^{n_1} + C_2 z^{n_2} + \frac{1}{K}\frac{\gamma}{1-\gamma} z^{-\frac{1-\gamma}{\gamma}} + \frac{\phi z}{r_z}. \tag{B.4}$$

Since

$$(1-\gamma)\mathcal{Q}(z) + \gamma z \mathcal{Q}'(z) = (1-\gamma+\gamma n_1) C_1 z^{n_1} + (1-\gamma+\gamma n_2) C_2 z^{n_2} + \frac{\phi z}{r_z},$$

we have

$$\begin{aligned} C_1 &= -\frac{1}{(1-\gamma+\gamma n_1)} \frac{(n_2-1)}{(n_2-n_1)} \left(\frac{\phi}{r_z} - 1\right) z_H^{1-n_1} < 0, \\ C_2 &= -\frac{1}{(1-\gamma+\gamma n_2)} \frac{(n_1-1)}{(n_1-n_2)} \left(\frac{\phi}{r_z} - 1\right) z_H^{1-n_2} > 0. \end{aligned} \tag{B.5}$$

It follows from $(1-\gamma)\mathcal{Q}(z_L) + \gamma z \mathcal{Q}'(z_L) = (1-\rho) z_L$ that

$$0 = \frac{n_2-1}{n_1-n_2}\xi^{n_1-1} + \frac{n_1-1}{n_2-n_1}\xi^{n_2-1} + 1 + \frac{r_z \rho}{\phi - r_z}, \tag{B.6}$$

where

$$\xi := \frac{z_H}{z_L} > 1. \tag{B.7}$$

Lemma 1. *There exists a unique $\xi > 1$ such that*

$$0 = \vartheta(\xi) := \frac{n_2-1}{n_1-n_2}\xi^{n_1-1} + \frac{n_1-1}{n_2-n_1}\xi^{n_2-1} + \frac{\phi - r_z(1-\rho)}{\phi - r_z}. \tag{B.8}$$

Proof. Note that

$$\vartheta(1) = \frac{\rho}{\phi - r_z} > 0 \;\text{ and }\; \vartheta(\infty) = -\infty. \tag{B.9}$$

Since for $\xi > 1$

$$\vartheta'(\xi) = \frac{(n_1-1)(n_2-1)}{(n_1-n_2)}\xi^{n_2-2}\left(\xi^{n_1-n_2} - 1\right) < 0, \tag{B.10}$$

we can conclude that there exists a unique $\xi > 1$ satisfying $\vartheta(\xi) = 0$. □

Since

$$\mathcal{Q}'(z_L) = 0 \;\text{ or }\; n_1 C_1 z_L^{n_1-1} + n_2 C_2 z_L^{n_2-1} - \frac{1}{K} z_L^{-\frac{1}{\gamma}} + \frac{\phi}{r_z} = 0, \tag{B.11}$$

we have

$$\begin{aligned} \frac{1}{K} z_L^{-\frac{1}{\gamma}} =& n_1 C_1 z_L^{n_1-1} + n_2 C_2 z_L^{n_2-1} + \frac{\phi}{r_z} \\ =& -\left(\frac{\phi}{r_z} - 1\right)\left(\frac{\xi^{n_1-1}}{(1-\gamma+\gamma n_1)}\frac{n_1(n_2-1)}{(n_2-n_1)} + \frac{\xi^{n_2-1}}{(1-\gamma+\gamma n_2)}\frac{n_2(n_1-1)}{(n_1-n_2)}\right) + \frac{\phi}{r_z}. \end{aligned} \tag{B.12}$$

Let us denote $\varrho(\xi)$ by

$$\varrho(\xi) := \frac{n_1(n_2-1)}{(n_1-n_2)}\xi^{n_1-1} - \frac{n_2(n_1-1)}{n_1-n_2}\xi^{n_2-1} + \frac{\phi}{\phi - r_z}. \tag{B.13}$$

By almost similar argument in the proof of Lemma 1, there exist a unique $\bar{\xi} > 1$ such that $\varrho(\bar{\xi}) = 0$. Then, we can set ρ^* by

$$\rho^* := -\frac{\phi - r_z}{r_z}\left(\frac{n_2-1}{n_1-n_2}\bar{\xi}^{n_1-1} + \frac{n_1-1}{n_2-n_1}\bar{\xi}^{n_2-1} + 1\right). \tag{B.14}$$

It is easy to check that

$$0 < \rho^* < 1$$

and

$$0 < \xi < \bar{\xi} \quad \text{for} \quad 0 < \rho < \rho^*. \tag{B.15}$$

Lemma 2. *$\mathcal{Q}(z)$ in (B.4) is strictly convex in $z \in (z_H, z_L)$ if and only if $0 < \rho < \rho^*$.*

Proof. Let us denote $\varphi(z)$ by

$$\varphi(z) := z\mathcal{Q}''(z) = n_1(n_1-1)C_1 z^{n_1-1} + n_2(n_2-1)C_2 z^{n_2-1} + \frac{1}{\gamma K}z^{-\frac{1}{\gamma}}. \tag{B.16}$$

Then, it follows from $C_1 < 0$ and $C_2 > 0$ that

$$\varphi'(z) = n_1(n_1-1)^2 C_1 z^{n_1-2} + n_2(n_2-1)^2 C_2 z^{m_2-1} - \frac{1}{\gamma^2}z^{-\frac{1}{\gamma}-1} < 0. \tag{B.17}$$

On the other hand, it follows from (B.12) that

$$\begin{aligned}\varphi(z_L) =& n_1(n_1-1)C_1 z_L^{n_1-1} + n_2(n_2-1)C_2 z_L^{n_2-1} + \frac{1}{\gamma K}z_L^{-\frac{1}{\gamma}} \\ =& n_1\left(n_1 - 1 + \frac{1}{\gamma}\right)C_1 z_L^{n_1-1} + n_2\left(n_2 - 1 + \frac{1}{\gamma}\right)C_2 z_L^{n_2-1} + \frac{1}{\gamma}\frac{\phi}{r_z} \\ =& \frac{1}{\gamma}\left(\frac{n_1(n_2-1)}{n_1-n_2}\left(\frac{\phi}{r_z}-1\right)\xi^{n_1-1} - \frac{n_2(n_1-1)}{n_1-n_2}\left(\frac{\phi}{r_z}-1\right)\xi^{n_2-1}\right) + \frac{1}{\gamma}\frac{\phi}{r_z}.\end{aligned} \tag{B.18}$$

Since

$$0 = \frac{n_2-1}{n_1-n_2}\xi^{n_1-1} + \frac{n_1-1}{n_2-n_1}\xi^{n_2-1} + \frac{\phi - r_z(1-\rho)}{\phi - r_z},$$

we can deduce that

$$\frac{d\xi}{d\rho} = -\frac{1}{\phi - r_z}\frac{n_1-n_2}{(n_1-1)(n_2-1)}\frac{1}{\xi^{n_1-2}(1-\xi^{n_2-n_1})} > 0. \tag{B.19}$$

This leads that

$$\frac{d\varphi(z_L)}{d\rho} = \left(\frac{\phi}{r_z}-1\right)\frac{(n_1-1)(n_2-1)}{n_1-n_2}\frac{\xi^{n_1-2}}{\gamma}\left(n_1 - n_2\xi^{n_2-n_1}\right)\frac{d\xi}{d\rho} < 0. \tag{B.20}$$

When $\rho = \rho^*$, it follows from the definition of ρ^* that

$$\varphi(z_L) = 0. \tag{B.21}$$

Hence, we can conclude that $\mathcal{Q}(z)$ is strictly convex in $z \in (z_H, z_L)$ if and only if $0 < \rho < \rho^*$. □

Lemma 3. *For $0 < \rho < \rho^*$, z_L given in (B.12) is a positive.*

Proof. Define the function $\zeta(\xi)$ by the right-hand side of the equation (B.12), i.e.,

$$\zeta(\xi) = -\left(\frac{\phi}{r_z} - 1\right)\left(\frac{\xi^{n_1-1}}{(1-\gamma+\gamma n_1)}\frac{n_1(n_2-1)}{(n_2-n_1)} + \frac{\xi^{n_2-1}}{(1-\gamma+\gamma n_2)}\frac{n_2(n_1-1)}{(n_1-n_2)}\right) + \frac{\phi}{r_z}. \tag{B.22}$$

Sine ξ is independent of $\gamma > 0$, we have

$$\frac{d\zeta(\xi)}{d\gamma} = \left(\frac{\phi}{r_z} - 1\right)\left(\frac{\xi^{1-n_1}}{(1-\gamma+\gamma n_1)^2}\frac{n_1(n_2-1)(n_1-1)}{(n_2-n_1)} + \frac{\xi^{n_2-1}}{(1-\gamma+\gamma n_2)^2}\frac{n_2(n_1-1)(n_2-1)}{(n_1-n_2)}\right) > 0. \tag{B.23}$$

Since

$$\begin{aligned}[\zeta(\xi)]_{\gamma=0} &= -\left(\frac{\phi}{r_z} - 1\right)\left(\xi^{n_1-1}\frac{n_1(n_2-1)}{(n_2-n_1)} + \xi^{n_2-1}\frac{n_2(n_1-1)}{(n_1-n_2)}\right) + \frac{\phi}{r_z} \\ &= \left(\frac{\phi}{r_z} - 1\right)\varrho(\xi) > \left(\frac{\phi}{r_z} - 1\right)\varrho(\bar{\xi}) = 0,\end{aligned} \tag{B.24}$$

where we have used the fact that $\varrho(\xi)$ is strictly decreasing in $0 < \xi < \bar{\xi}$.

Hence, we deduce that $\zeta(\xi)$ is positive for $0 < \rho < \rho^*$. □

For $\rho \geq \rho^*$, let us the following free boundary problem:

$$\begin{cases} \mathcal{L}_z\mathcal{Q} + \tilde{u}(z) + \phi z = 0 \ \text{ for } \ \hat{z}_H < z < \hat{z}_L, \\ (1-\gamma)\mathcal{Q}(z_H) + \gamma z_H\mathcal{Q}'(\hat{z}_H) = \hat{z}_H, \quad \mathcal{Q}'(z_H) + \gamma z_H\mathcal{Q}''(\hat{z}_H) = 1, \\ \mathcal{Q}'(\hat{z}_L) = 0, \quad \mathcal{Q}''(\hat{z}_L) = 0. \end{cases} \tag{B.25}$$

For $z_H < z < z_L$, a general solution of $\mathcal{Q}(z)$ is given by

$$\mathcal{Q}(z) = \hat{C}_1 z^{n_1} + \hat{C}_2 z^{n_2} + \frac{1}{K}\frac{\gamma}{1-\gamma}z^{-\frac{1-\gamma}{\gamma}} + \frac{\phi z}{r_z}. \tag{B.26}$$

By the same argument in the previous case for $0 < \rho < \rho^*$, we have

$$\begin{aligned}\hat{C}_1 &= -\frac{1}{(1-\gamma+\gamma n_1)}\frac{(n_2-1)}{(n_2-n_1)}\left(\frac{\phi}{r_z} - 1\right)\hat{z}_H^{1-n_1} < 0, \\ \hat{C}_2 &= -\frac{1}{(1-\gamma+\gamma n_2)}\frac{(n_1-1)}{(n_1-n_2)}\left(\frac{\phi}{r_z} - 1\right)\hat{z}_H^{1-n_2} > 0.\end{aligned} \tag{B.27}$$

Since

$$\mathcal{Q}'(\hat{z}_L) + \gamma\hat{z}_L\mathcal{Q}''(\hat{z}_L) = 0,$$

we deduce that

$$0 = \hat{\vartheta}(\hat{\xi}) := -\frac{n_1(n_2-1)}{(n_2-n_1)}\hat{\xi}^{n_1-1} - \frac{n_2(n_1-1)}{(n_1-n_2)}\hat{\xi}^{n_2-1} + \frac{\phi}{\phi - r_z}, \tag{B.28}$$

where

$$\hat{\xi} := \frac{\hat{z}_L}{\hat{z}_H} > 1. \tag{B.29}$$

We can easily verify that there exists a unique $\hat{\xi} > 1$ such that $\hat{\vartheta}(\hat{\xi}) = 0$. Moreover, it follows from $\mathcal{Q}'(\hat{z}_L) = 0$ that

$$\frac{1}{K}\hat{z}_L^{-\frac{1}{\gamma}} = -\left(\frac{\phi}{r_z} - 1\right)\left(\frac{\hat{\xi}^{n_1-1}}{(1-\gamma+\gamma n_1)}\frac{n_1(n_2-1)}{(n_2-n_1)} + \frac{\hat{\xi}^{n_2-1}}{(1-\gamma+\gamma n_2)}\frac{n_2(n_1-1)}{(n_1-n_2)}\right) + \frac{\phi}{r_z}. \tag{B.30}$$

Note that $z_L > 0$.

Lemma 4. *$\mathcal{Q}(z)$ in (B.26) is strictly convex in $z \in (\hat{z}_H, \hat{z}_L)$.*

Proof. Note that for $\hat{z}_H < z < \hat{z}_L$

$$\mathcal{Q}''(z) = n_1(n_1-1)\hat{C}_1 z^{n_1-2} + n_2(n_2-1)\hat{C}_2 z^{n_2-2} + \frac{1}{\gamma K} z^{-\frac{1}{\gamma}-1}. \tag{B.31}$$

Let us temporarily denote $\hat{\varphi}(z)$ by

$$\hat{\varphi}(z) := z\mathcal{Q}''(z) = n_1(n_1-1)\hat{C}_1 z^{n_1-1} + n_2(n_2-1)\hat{C}_2 z^{n_2-1} + \frac{1}{\gamma K} z^{-\frac{1}{\gamma}}. \tag{B.32}$$

Since

$$\hat{\varphi}'(z) = n_1(n_1-1)^2\hat{C}_1 z^{n_1-2} + n_2(n_2-1)^2\hat{C}_2 z^{n_2-2} - \frac{1}{\gamma^2 K} z^{-\frac{1}{\gamma}-1} < 0, \tag{B.33}$$

it follows from $\hat{\varphi}(\hat{z}_L) = \hat{z}_L \mathcal{Q}''(\hat{z}_L) = 0$ that

$$\mathcal{Q}''(z) > 0 \quad \text{for} \quad \hat{z}_H < z < \hat{z}_L.$$

□

Sum up, we derive the strictly convex function $\mathcal{Q}(z)$ in the inaction region as follows:

$$\mathcal{Q}(z) = \begin{cases} \hat{C}_1 z^{n_1} + \hat{C}_2 z^{n_2} + \dfrac{\gamma}{1-\gamma} z^{-\frac{1-\gamma}{\gamma}} + \dfrac{\phi z}{r_z} & \text{for} \ \ \rho \geq \rho^*, \\ C_1 z^{n_1} + C_2 z^{n_2} + \dfrac{\gamma}{1-\gamma} z^{-\frac{1-\gamma}{\gamma}} + \dfrac{\phi z}{r_z} & \text{for} \ \ 0 < \rho < \rho^*. \end{cases} \tag{B.34}$$

By utilizing the standard verification arguments, We can easily show that $\mathcal{Q}(z)$ in (B.34) satisfies the HJB equation (B.2), and

$$\begin{aligned} J(y,k,s) = \omega(ks)^{1-\gamma}\mathcal{Q}\left(\frac{y(ks)^\gamma}{\omega}\right) &= \sup_{\{\mathcal{G}_t^+,\mathcal{G}_t^-\}} \inf_{\{D_t\}} \mathcal{J}(y,k,s;\{D_t\}_{t=0}^\infty, \{\mathcal{G}_t^+,\mathcal{G}_t^-\}_{t=0}^\infty) \\ &= \inf_{\{D_t\}} \sup_{\{\mathcal{G}_t^+,\mathcal{G}_t^-\}} \mathcal{J}(y,k,s;\{D_t\}_{t=0}^\infty, \{\mathcal{G}_t^+,\mathcal{G}_t^-\}_{t=0}^\infty). \end{aligned}$$

For $0 < \omega < 1$, the procedures and logic for deriving the solution are almost similar to the case when $\omega = 1$. Thus, we omit the detailed proofs.

B.3. Proof of Proposition 3

For simplicity, we first consider the case when $\omega = 1$.

Note that for $0 < \rho < \rho^*$

$$\begin{aligned}\mathcal{W}_H = -\mathcal{Q}'(z_H) = & -n_1 C_1 z_H^{n_1-1} - n_2 C_2 z_H^{n_2-1} + \frac{1}{K} z_H^{-\frac{1}{\gamma}} - \frac{\phi}{r_z} \\ = & \frac{n_1(n_2-1)}{(1-\gamma+\gamma n_1)(n_2-n_1)}\left(\frac{\phi}{r_z}-1\right) + \frac{n_2(n_1-1)}{(1-\gamma+\gamma n_2)(n_1-n_2)}\left(\frac{\phi}{r_z}-1\right) \\ & + \frac{1}{K} z_H^{-\frac{1}{\gamma}} - \frac{\phi}{r_z}\end{aligned} \tag{B.35}$$

and for $\rho \geq \rho^*$

$$\begin{aligned}\mathcal{W}_H = -\mathcal{Q}'(\hat{z}_H) = & \frac{n_1(n_2-1)}{(1-\gamma+\gamma n_1)(n_2-n_1)}\left(\frac{\phi}{r_z}-1\right) + \frac{n_2(n_1-1)}{(1-\gamma+\gamma n_2)(n_1-n_2)}\left(\frac{\phi}{r_z}-1\right) \\ & + \frac{1}{K} \hat{z}_H^{-\frac{1}{\gamma}} - \frac{\phi}{r_z}.\end{aligned} \tag{B.36}$$

It follows from the algebraic equations (B.8) and (B.28) that

$$\frac{\partial \xi}{\partial \rho} = -\frac{1}{\phi - r_z} \frac{n_1 - n_2}{(n_1-1)(n_2-1)} \frac{1}{\xi^{n_1-2} - \xi^{n_2-2}} > 0. \tag{B.37}$$

From (B.12),

$$\frac{1}{K} z_H^{-\frac{1}{\gamma}} = -\left(\frac{\phi}{r_z}-1\right)\left(\frac{\xi^{n_1-1+\frac{1}{\gamma}}}{(1-\gamma+\gamma n_1)} \frac{n_1(n_2-1)}{(n_2-n_1)} + \frac{\xi^{n_2-1+\frac{1}{\gamma}}}{(1-\gamma+\gamma n_2)} \frac{n_2(n_1-1)}{(n_1-n_2)}\right) + \frac{\phi}{r_z}\xi^{\frac{1}{\gamma}}. \tag{B.38}$$

This implies that for $0 < \rho < \rho^*$

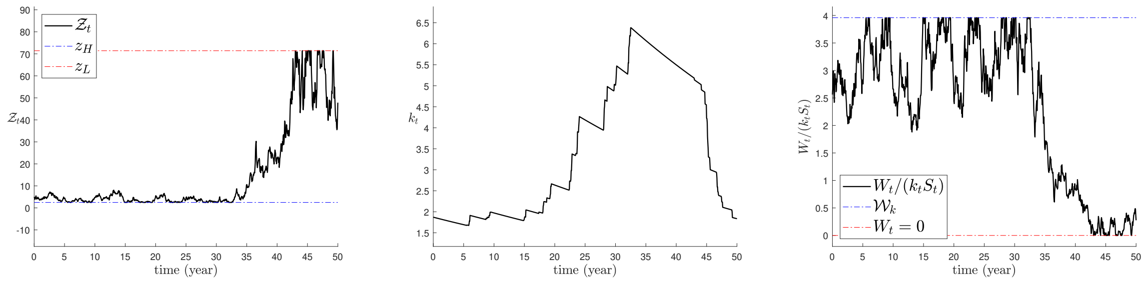
$$\begin{aligned}\frac{\partial \mathcal{W}_H}{\partial \rho} = & \frac{d}{d\rho}\left(\frac{1}{K} z_H^{-\frac{1}{\gamma}}\right) \\ = & \left(\frac{\phi}{r_z}-1\right)\left(\xi^{n_1-1}\frac{n_1(n_2-1)}{(n_1-n_2)} - \xi^{n_2-1}\frac{n_2(n_1-1)}{(n_1-n_2)} + \frac{\phi}{\phi - r_z}\right)\xi^{\frac{1}{\gamma}-1} \\ = & \left(\frac{\phi}{r_z}-1\right)\xi^{\frac{1}{\gamma}-1}\varrho(\xi) > 0,\end{aligned} \tag{B.39}$$

where we have used the fact that $\varrho(\xi)$ is strictly decreasing in $\xi \in (1, \bar{\xi})$ and $\varrho(\bar{\xi}) = 0$.

Similarly, for any $\omega \in (0, 1]$, we can show that

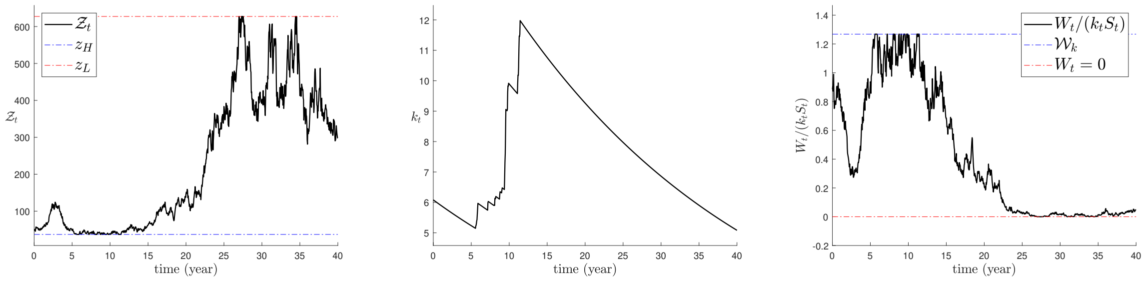
$$\frac{\partial \mathcal{W}_H}{\partial \rho} > 0, \quad \frac{\partial \mathcal{W}_H}{\partial \phi} > 0, \quad \text{and} \quad \frac{\partial \mathcal{W}_H}{\partial \omega} > 0. \tag{B.40}$$

C. Additional Figures



(a) $\mathcal{Z}_t$ (b) k_t^* (c) $\frac{W_t}{m_t}$

Figure 13: Simulation path $\mathcal{Z}_t$, k_t, and $\frac{W_t}{m_t}$ with $\rho = 0.01$. The parameter set is given by $\gamma = 2, \beta = 0.05, \mu = 0.07, \sigma = 0.2, r = 0.03, \omega = 0.3, \phi = 0.02$.



(a) $\mathcal{Z}_t$ (b) k_t^* (c) $\frac{W_t}{m_t}$

Figure 14: Simulation path $\mathcal{Z}_t$, k_t, and $\frac{W_t}{m_t}$ with $\rho = 0.3$. The parameter set is given by $\gamma = 3/2, \beta = 0.05, \mu = 0.07, \sigma = 0.2, r = 0.03, \omega = 0.3, \phi = 0.02$.

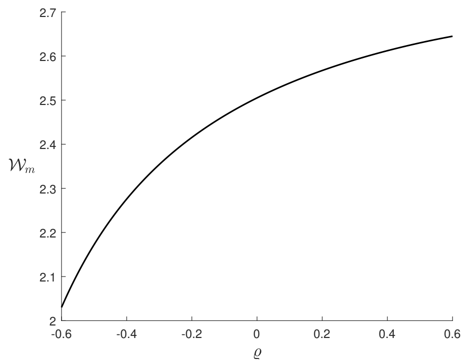


Figure 15: $\mathcal{W}_H$ as a function of ϱ. The parameter set is given by $\gamma = 3, \beta = 0.07, \mu = 0.08, \mu_p = 0.07, \sigma = 0.3, \sigma_p = 0.2, r = 0.03, \phi = 0.02, \rho = 0.02, \omega = 0.5$

Figure 13 and 14 illustrate the simulation path of 50 years for $\mathcal{Z}_t$, k_t^*, and $\frac{W_t}{m_t}$. The boundary z_L (or z_H) corresponds to the wealth level of the minimum wealth to staking ratio $W_t = 0$ (or the maximum wealth to staking ratio $\mathcal{W}_m$). For the case with a small ρ (Figure 13), whenever the dual variable hits z_H (or z_L), the number of staked tokens k_t decreases (or increases). When ρ is large, however, even when the dual variable $\mathcal{Z}_t$ hits the boundary z_L, the staking amount k_t is unchanged as shown in Figure 14. Instead, the shadow price process D_t decreases whenever $\mathcal{Z}_t$ hits z_L in this case.

Figure 15 shows how the correlation between the stock market and the cryptocurrency market affects the inaction interval of the staking token. It increases as the correlation increases.